Software Polish

Rick Groszkiewicz
2964 Nestle Creek Drive
Marietta, GA 30062-4857

Voice/fax (770) 971-8913

email: rickg@softwarepolish.com

SPRING 1994 EA-1A
EXAM SOLUTIONS

Revision History:

02/23/99	Enhanced problems 5, 12, 15, 19	added faster method of solution
	Corrected problem 17	several typos in solution
02/19/00	Corrected problem 13	added faster method of solution
	Corrected problem 18	final answer was incorrect value

Spring 1994 EA-1A Solutions

1 For bond problems, you have numerous price formulas to choose from. For this problem, the "standard" formula works well.

$$P = Fr \cdot a_{\overline{n}|i} + K \quad (\text{"FRANK"})$$

$P = .90 \qquad K = C \cdot v^n$

$F = 1.00 \qquad \quad = 1.00\, v^n$

$i = 6\%$

$$.90 = (1.0)r\,(a_{\overline{15}|6\%}) + (1.0)v^{15}$$

$$.9 - (1.06)^{-15} = r \cdot a_{\overline{15}|.06}$$

$$r = \frac{.9 - (1.06)^{-15}}{a_{\overline{15}|.06}}$$

$$= \frac{.9 - .4173}{9.7122}$$

$$= .0497$$

D

Spring 1994 EA-1A Solutions

2 You need to know some identities to solve for i based on the given items.

$N_x = \Sigma D_x \qquad S_x = \Sigma N_x$

$M_x = \Sigma C_x \qquad R_x = \Sigma M_x$

$M_x = D_x - d \cdot N_x \qquad R_x = N_x - d \cdot S_x$

$$= (S_x - S_{x+1}) - d \cdot S_x$$

$$= S_x(1-d) - S_{x+1}$$

$$= v \cdot S_x - S_{x+1}$$

$$\therefore \quad R_x + S_{x+1} = v \cdot S_x$$

$$\frac{R_x + S_{x+1}}{S_x} = v$$

$$v = \frac{69{,}651{,}728 + 6{,}313{,}352{,}030}{6{,}765{,}983{,}984}$$

$$= .9434$$

$$1+i = 1.0600$$

$$i = 6.0\%$$

C

Spring 1994 EA-1A Solutions

3 This problem requires you to write down a series of probability terms. One trick to the problem is that A and B can die in the same year, and the chance that A dies before B is 50% of that combined probability of death.

<u>A dies in 1994</u>

B dies in 1994: $.5(q_{90})(q_{90}) = .5(.18)(.18)$

" " " 1995: $q_{90}(p_{90} \cdot q_{91}) = .18(.82)(.20)$

" " " 1996: $q_{90}(p_{90})(p_{91} \cdot q_{92}) = .18(.82)(.80)(22)$

<u>A dies in 1995</u>

B dies in 1995: $(p_{90} \cdot p_{90}).5(q_{91} \cdot q_{91}) = (.82)^2.5(.20)^2$

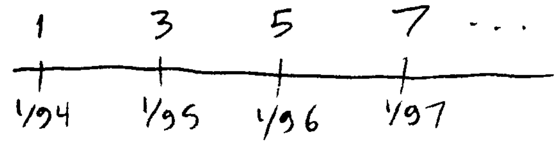
B dies in 1996: $(p_{90} \cdot p_{90})q_{91}(p_{91} \cdot q_{92}) = (.82)^2.20(.80).22$

Total probability $= .01620 + .02952 + .02598 + .01345 + .02367$
$= .1088$

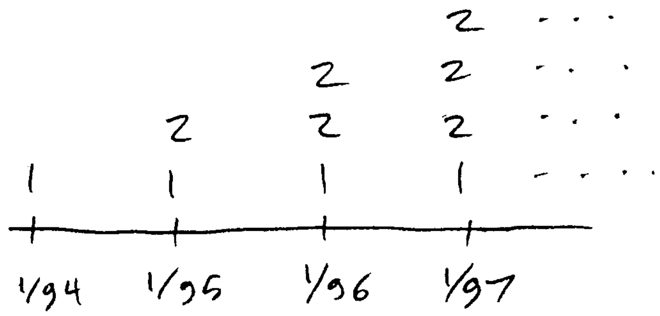
B

Spring 1994 EA-1A Solutions

4 The time line diagram for the increasing perpetuity looks like this:



This can be redrawn as a series of payments



let $P = 1 + v + v^2 + \dots$ = PV of perpetuity of \$1

$25P = 1 + 3v + 5v^2 + 7v^3 + \dots$ = PV of increasing perpetuity

$25P = P$: layer of \$1 perpetuity starting 1/94

$+ 2vP$: " " \$2 " " 1/95

$+ 2v^2P$: " " \$2 " " 1/96

$+ 2v^3P$: " " \$2 " " 1/97

$+ \dots$

$$25P = 2P(1 + v + v^2 + \dots) - P$$

$$25 = 2\left(\frac{1 - v^\infty}{iv}\right) - 1 = \frac{2}{1-v} - 1$$

$$26 = \frac{2}{1-v} \Rightarrow 1 - v = \frac{2}{26} \Rightarrow v = \frac{24}{26} \Rightarrow i = .0833$$

C

5 There are several ways to work these identity questions. This problem is unusual in that you are given no values for A, B, or i. By using the information given that $i(a_{\overline{A|}})(a_{\overline{B|}}) = 1.67$, you can get to the answer relatively quickly.

First you must write down the information on each of the three payment options. The present value of all loan payments must equal 1000:

<u>Option</u>

Option		
1	$1000 = 96.34\,(a_{\overline{A+B|}})$	$\Rightarrow a_{\overline{A+B|}} = 10.3799$
2	$1000 = 129.51\,(a_{\overline{B|}})$	$\Rightarrow a_{\overline{B|}} = 7.7214$
3	$1000 = P\,(a_{\overline{A+B|}} + v^A a_{\overline{B|}})$	

$$
\begin{aligned}
1.67 &= i\,(a_{\overline{A|}})(a_{\overline{B|}}) \\
&= (1-v^A)\,a_{\overline{B|}} \\
&= (1-v^A)\,7.7214 \\
v^A &= 1-(1.67/7.7214) = .7837
\end{aligned}
$$

Now you can substitute this value into the information for option 3, and solve for P

$$
\begin{aligned}
1000 &= P\,(10.3799 + .7837(7.7214)) \\
P &= 60.86
\end{aligned}
$$

C

Another solution is shown on the next page, which is a little different in how it uses the information that $1.67 = i\,(a_{\overline{A|}})(a_{\overline{B|}})$

Spring 1994 EA-1A Solutions

(5) The first step is to write down relationships for the three series of payments

Option 1: $1000 = 96.34\, a_{\overline{A+B}|}$

Option 2: $1000 = 129.51\, a_{\overline{B}|}$

Option 3: $1000 = P\left(a_{\overline{A+B}|} + v^{A} \cdot a_{\overline{B}|}\right)$

$= 2P\, a_{\overline{A+B}|} - P \cdot a_{\overline{A}|}$

From option 1, you can calculate $a_{\overline{A+B}|} = 10.3799$.
From option 2, you can calculate $a_{\overline{B}|} = 7.7214$.

The trick to the problem is to rewrite the given information in a more usable form:

$$1.67 = i\,(a_{\overline{A}|})(a_{\overline{B}|}) = i\,\frac{(1-v^{A})}{i}\frac{(1-v^{B})}{i}$$

$$= \frac{1-v^{A}-v^{B}+v^{A+B}}{i}$$

$$= a_{\overline{A}|} + a_{\overline{B}|} - a_{\overline{A+B}|}$$

$$\therefore a_{\overline{A}|} = 1.67 - a_{\overline{B}|} + a_{\overline{A+B}|}$$

$$= 4.3285$$

From option 3, you have

$$1000 = 2P(a_{\overline{A+B}|}) - P(a_{\overline{A}|})$$

$$\frac{1000}{2(a_{\overline{A+B}|}) - a_{\overline{A}|}} = P$$

$$\therefore P = 60.86$$

C

Spring 1994 EA-1A Solutions

6 For a double decrement table, there is a standard approximation you can use to calculate single decrement rates in terms of the probabilities:

$$q'^{(d)}_x \doteq \frac{q^{(d)}_x}{1-\frac{1}{2}q^{(w)}_x} = \frac{q^{(T)}_x - q^{(w)}_x}{1-\frac{1}{2}q^{(w)}_x}$$

$$.022 \doteq \frac{.3 - q^{(w)}_x}{1-.5\,q^{(w)}_x}$$

$$.022 - .011\,q^{(w)}_x \doteq .3 - q^{(w)}_x$$

$$.989\,q^{(w)}_x \doteq .278$$

$$q^{(w)}_x \doteq .2811$$

D

The other way to work this problem is to use Bowers' formula:

$$p'^{(d)}_x = \left[p^{(T)}_x\right]^{q^{(d)}_x / q^{(T)}_x}$$

Let $z = q^{(d)}_x / q^{(T)}_x$

$$p'^{(d)}_x = \left[p^{(T)}_x\right]^z$$

$$1 - .022 = [1 - .30]^z$$

$$\log .978 = z(\log .70) \Rightarrow z = \frac{\log .978}{\log .700} = .0624$$

$$.0624 = q^{(d)}_x / q^{(T)}_x \Rightarrow q^{(d)}_x = .0624(.30) = .0187$$

$$q^{(w)}_x = q^{(T)}_x - q^{(d)}_x = .2813$$

Spring 1994 EA-1A Solutions

7 The definition of the complete expectation of life is

$$\mathring{e}_x \doteq e_x + \tfrac{1}{2}$$

Based on the l_x formula given, you can transform this into DeMoivre's law, where $l_x = \omega - x$ and $\omega = 100$. The formula for an annuity immediate under DeMoivre's law is

$$a_x = \frac{n - \ddot{a}_{\overline{n}|i}}{ni} \quad \text{where } n = \omega - x$$

We can't directly use this with a zero interest rate to determine expectation of life. Instead, write down an expression in simplest terms

$$
\begin{aligned}
e_x &= p_x + {}_2p_x + \ldots \\
&= \frac{l_{x+1}}{l_x} + \frac{l_{x+2}}{l_x} + \cdots \frac{l_{\omega-1}}{l_x} + \frac{l_\omega}{l_x} \\
&= \frac{\omega-x-1}{\omega-x} + \frac{\omega-x-2}{\omega-x} + \cdots + \frac{\omega-x-(\omega-x-1)}{\omega-x} + \emptyset \\
&= 1-\frac{1}{\omega-x} + 1-\frac{2}{\omega-x} + \cdots + 1-\frac{\omega-x-1}{\omega-x} \\
&= (\omega-x-1) - \frac{1}{\omega-x}(1+2+\ldots+\omega-x-1) \\
&= \omega-x-1 - \frac{\omega-x-1}{2}\left(\frac{\omega-x}{\omega-x}\right) \\
&= \frac{\omega-x-1}{2}
\end{aligned}
$$

$e_{50} = \frac{49}{2} = 24.5 \qquad \mathring{e}_{50} \doteq 24.5 + .5 = 25.0$ (E)

Spring 1994 EA-1A Solutions

8 There are two way to work this problem. The quicker method uses knowledge of insurance identities:

$$
\begin{aligned}
{}_{18}V_{45:\overline{20}|} &= A_{63:\overline{2}|} - P_{45:\overline{20}|}\left(\ddot{a}_{63:\overline{2}|}\right) \\
&= 1 - d\left(\ddot{a}_{63:\overline{2}|}\right) - \frac{A_{45:\overline{20}|}}{\ddot{a}_{45:\overline{20}|}}\left(\ddot{a}_{63:\overline{2}|}\right) \\
&= 1 - d\left(\ddot{a}_{63:\overline{2}|}\right) - \left(\frac{1 - d\,\ddot{a}_{45:\overline{20}|}}{\ddot{a}_{45:\overline{20}|}}\right)\ddot{a}_{63:\overline{2}|} \\
&= 1 - \frac{\ddot{a}_{63:\overline{2}|}}{\ddot{a}_{45:\overline{20}|}}
\end{aligned}
$$

$$\ddot{a}_{63:\overline{2}|} = 1 + v\,p_{63} = 1 + \frac{.9813}{1.07} = 1.9171$$

$$\therefore\ {}_{18}V_{45:\overline{20}|} = 1 - \frac{1.9171}{10.84} = .8231$$

(B)

The other method requires you to calculate values for $A_{63:\overline{2}|}$ and $P_{45:\overline{20}|}$ in addition to the value for $\ddot{a}_{63:\overline{2}|}$

$$
\begin{aligned}
P_{45:\overline{20}|} &= \frac{A_{45:\overline{20}|}}{\ddot{a}_{45:\overline{20}|}} = \frac{1 - d\,\ddot{a}_{45:\overline{20}|}}{\ddot{a}_{45:\overline{20}|}} = \frac{1}{\ddot{a}_{45:\overline{20}|}} - d \\
&= \frac{1}{10.84} - \frac{.07}{1.07} = .02683
\end{aligned}
$$

$$
\begin{aligned}
A_{63:\overline{2}|} &= v\,q_{63} + p_{63}(v^2)(p_{64} + q_{64}) \\
&= \frac{.0187}{1.07} + \frac{.9813}{(1.07)^2}(1) = .8746
\end{aligned}
$$

$${}_{18}V_{45:\overline{20}|} = .8746 - .02683(1.9171) = .8232$$

Spring 1994 EA-1A Solutions

9 The first step is to draw a time diagram for the series of payments in this problem:

$$\begin{array}{ccccc} & P & P(1.04) & & P(1.04)^{29} + 1{,}000{,}000(1.02)^{30} \\ \otimes & | & | & \cdots & | \\ 1\text{-}94 & 1\text{-}95 & 1\text{-}96 & & 1\text{-}1\text{-}24 \end{array}$$

NOTE: $(1.05)^2 = 1.1025$

$$1{,}000{,}000 = 1{,}000{,}000\left(\frac{1.02}{1.1025}\right)^{30} + \frac{P}{1.1025} + \frac{P(1.04)}{(1.1025)^2} + \dots + \frac{P(1.04)^{29}}{(1.1025)^{30}}$$

$$= 1{,}000{,}000\left(\frac{1.02}{1.1025}\right)^{30} + \frac{P}{1.1025}\left[1 + \frac{1.04}{1.1025} + \dots + \left(\frac{1.04}{1.1025}\right)^{29}\right]$$

$$= 1{,}000{,}000\left(\frac{1.02}{1.1025}\right)^{30} + \frac{P}{1.1025}\ \ddot{a}_{\overline{30}|6.01\%}$$

$$P = 1{,}000{,}000\left(1 - \left(\frac{1.02}{1.1025}\right)^{30}\right)\frac{1.1025}{\ddot{a}_{\overline{30}|6.01\%}}$$

$$= 68{,}298$$

Final payment is $P(1.04)^{29} + 1{,}000{,}000(1.02)^{30}$

$= 212{,}999 + 1{,}811{,}362$

$= 2{,}024{,}360$

D

Spring 1994 EA-1A Solutions

10 This problem requires that you know the formula for an annuity payable quarterly:

$$\ddot{a}_x^{(m)} \doteq \ddot{a}_x - \frac{m-1}{2m}$$

$$\ddot{a}_x^{(4)} \doteq \ddot{a}_x - \frac{3}{8}$$

Let P be the annual premium, and Q be the annual amount of premium paid quarterly:

$$1000 A_x = P \cdot \ddot{a}_x$$
$$1000 \frac{A_x}{\ddot{a}_x} = P = \frac{700}{5} = 140$$

$$1000 A_x = Q \cdot \ddot{a}_x^{(4)}$$
$$1000 \frac{A_x}{\ddot{a}_x^{(4)}} = Q = \frac{700}{5 - 3/8} = 151.35$$

For a $1000 life insurance policy, the increase in premium to pay on a quarterly basis is 11.35

(C)

Spring 1994 EA-1A Solutions

11 You must develop market values before and after each cash flow to calculate the time weighted yield. The following time diagram shows the cash flows at each quarterly date. Below the diagram are the market values

	1-1	4-1	7-1	10-1	12-31
		+ 10,000 − 10,000	+30,000 −10,000	+50,000 −10,000	
MV_B		150,000	100,000	160,000	200,000
given → MV_A	100,000	150,000	120,000	200,000	200,000

At 7-1, the 100,000 is calculated as 120,000 (given) after the cash flows, less the 20,000 cash flows at 6-30.

The time weighted return is calculated by multiplying the ratios of market values from quarter to quarter

$$1+i_t = \underset{1^{st}Q}{\left(\frac{150}{100}\right)} \underset{2^{nd}Q}{\left(\frac{100}{150}\right)} \underset{3^{rd}Q}{\left(\frac{160}{120}\right)} \underset{4^{th}Q}{\left(\frac{200}{200}\right)} = \frac{160}{120} = 1.3333$$

(next page)

(11) continued

For the dollar-weighted return, you can use a similar time diagram:

1-1	4-1	7-1	10-1	12-31
100,000		20,000	40,000	
				200,000

$$100{,}000\,(1+i) + 20{,}000\left(1+\tfrac{i}{2}\right) + 40{,}000\left(1+\tfrac{i}{4}\right) = 200{,}000$$

$$i\left[100{,}000 + \tfrac{6}{12}(20{,}000) + \tfrac{3}{12}(40{,}000)\right] = 200{,}000 - 160{,}000$$

$$i_d = \frac{40{,}000}{100{,}000 + \frac{1}{2}(20{,}000) + \frac{1}{4}(40{,}000)}$$

$=$

You could write this formula down by looking at the time diagram above. The total amount of interest of 40,000 is the difference between the beginning and ending market values and the cash flows.

The denominator of the fraction represents the weight or exposure, which is each cash flow multiplied by the time to the end of the year.

$$i_d = \frac{40}{120} = \frac{1}{3} = 33.33\%$$

$\therefore \Delta$ between i_d and i_t is zero (A)

Spring 1994 EA-1A   Solution added 2/23/99

12 As with most "identity type" problems, there are many ways to work them. Your approach to the problem depends on your ability to find the specific identity or relationship that gives you the quickest route to the answer.

The fastest solution to this problem is based on simple identities for C_x and D_x:

$$C_x = v^{x+1} d_x$$
$$D_x = v^x \ell_x$$
$$\frac{C_x}{D_x} = v q_x \qquad \frac{D_{x+1}}{D_x} = v p_x$$
$$\frac{C_x / D_x}{D_{x+1}/D_x} = \frac{v q_x}{v p_x}$$
$$\frac{C_x}{D_{x+1}} = \frac{q_x}{p_x}$$
$$C_x(1-q_x) = q_x D_{x+1}$$
$$q_x = \frac{C_x}{C_x + D_{x+1}}$$

You need to calculate $C_{64} = M_{64} - M_{65}$
$= 716{,}531 - 686{,}751$
$= 29{,}780$

$$q_{64} = \frac{C_{64}}{C_{64} + D_{65}} = \frac{29{,}780}{29{,}780 + 995{,}688}$$
$= .0290$

E

The next page shows another method of solution which requires you to solve for the interest rate.

Spring 1994 EA-1A Solutions

(12) This is a surprisingly difficult problem. If you don't find the easy way to derive the interest rate, you will have two equations in two unknowns, which will be i and p_{64}. Solving these is very messy.

The fastest solution starts with C_x:

$$
\begin{aligned}
C_x &= M_x - M_{x+1} \\
&= v^{x+1} d_x \\
&= v^{x+1}(l_x - l_{x+1}) \\
&= v^{x+1} l_x - v^{x+1} l_{x+1} \\
&= v D_x - D_{x+1}
\end{aligned}
$$

$$\therefore \; vD_x - D_{x+1} = M_x - M_{x+1}$$

$$v = \frac{M_x - M_{x+1} + D_{x+1}}{D_x} = \frac{716{,}531 - 686{,}751 + 995{,}688}{1{,}056{,}232}$$

$$= .9709$$

$$\therefore \; i = 3.0\%$$

$$v\,p_{64} = D_{65} / D_{64}$$

$$
\begin{aligned}
1 - q_{64} &= (1+i)\,D_{65}/D_{64} \\
q_{64} &= 1 - \frac{(1+i)D_{65}}{D_{64}} \\
&= 1 - \frac{1.03(995{,}688)}{1{,}056{,}232} \\
&= .0290
\end{aligned}
$$

(E)

Spring 1994 EA-1A Solutions

13 Based on the small amount of information you are given, this must be primarily an algebra problem. There are many ways to arrive the correct answer. This pages shows the clearest method of solution, with a different method on the next page.

The first key to working this problem is correctly writing down expressions for the present value of each annuity:

A: $10{,}000(\ddot{a}_{\overline{10}|i})$

B: $7{,}600(\ddot{a}_{\overline{10}|i} + {}_{10|}\ddot{a}_x)$

C: $P(\ddot{a}_{\overline{10}|i}) + 1.10(P)({}_{10|}\ddot{a}_x)$

Since the annuities are actuarially equivalent, all of the present values must be equal

$$10{,}000\,\ddot{a}_{\overline{10}|i} = 7{,}600\,(\ddot{a}_{\overline{10}|i} + {}_{10|}\ddot{a}_x)$$
$$10{,}000\,\ddot{a}_{\overline{10}|i} = P(\ddot{a}_{\overline{10}|i}) + 1.10(P)({}_{10|}\ddot{a}_x)$$

Now divide both equations by $\ddot{a}_{\overline{10}|i}$

$$10{,}000 = 7.600\,(1 + ({}_{10|}\ddot{a}_x / \ddot{a}_{\overline{10}|i}))$$
$$10{,}000 = P + 1.10(P)({}_{10|}\ddot{a}_x / \ddot{a}_{\overline{10}|i})$$

Now you can solve for the value of the ratio of two annuities from the first equation, substitute that value in the second equation, and solve for P:

$$(10{,}000/7{,}600) - 1 = {}_{10|}\ddot{a}_x / \ddot{a}_{\overline{10}|i} = .3158$$
$$P = 10{,}000/(1 + 1.10(.3158)) = 7{,}422 \quad \text{D}$$

Spring 1994 EA-1A Solutions

(13) (Continued - Alternate solution)

The first step is to write down the present value of each of the three optional forms of benefit payment:

Option A: $10{,}000\ \ddot{a}_{\overline{10}|i}$

Option B: $7{,}600\ (\ddot{a}_{\overline{10}|i} + {}_{10|}\ddot{a}_x)$

Option C: $P\ (\ddot{a}_{\overline{10}|i}) + 1.1P\ ({}_{10|}\ddot{a}_x)$

Next, you should equate the values under options B and C to option A. You can simplify the process if you replace the $\ddot{a}_{\overline{10}|i}$ with C (for certain annuity) and ${}_{10|}\ddot{a}_x$ with L (for deferred life annuity):

A=B:

$$10\,000\,C = 7600\,C + 7600\,L$$
$$2400\,C = 7600\,L$$
$$\frac{C}{L} = \frac{76}{24}$$

A=C:

$$10000\,C = P\cdot C + 1.1P\cdot L$$
$$(10000-P)\,C = 1.1P\cdot L$$
$$\frac{C}{L} = \frac{1.1P}{10\,000-P}$$

$$\frac{1.1P}{10{,}000-P} = \frac{76}{24}$$
$$26.4P = 760{,}000 - 76P$$
$$102.4P = 760{,}000$$
$$P = 7422$$

D

Spring 1994 EA-1A Solutions

14 There are two approaches to work this problem, which should give the same answer. I'll show the "safe" solution first:

At 1-1-94, you know that $100{,}000 = 1{,}000\, a_{\overline{n}|.5\%}$

At 1-1-95, the outstanding loan balance is

$$\begin{aligned} \text{O/S loan} &= 100{,}000(1.005)^{12} - s_{\overline{12}|.5\%}(1000) \\ &= 106{,}168 - 12{,}336 \\ &= 93{,}832 \end{aligned}$$

For renegotiated loan, you have $93{,}832 = P \cdot a_{\overline{360-12}|.5\%}$

$$P = 93{,}832 / a_{\overline{348}|.5\%} = 569.57$$
$$\Delta = 1000 - 569.57 = 430.43$$

A

The alternate approach is to solve for the term of the original loan. Then the O/S loan balance at 1-1-95 can be calculated directly.

$$100{,}000 = 1000\, a_{\overline{n}|.5\%} \Rightarrow n = 139$$
$$1000(a_{\overline{139}|}) = 100{,}012 \quad \text{(not quite exact)}$$

$$\text{1-1-95 O/S loan balance} = 1000(a_{\overline{127}|.5\%}) = 93{,}845$$
$$P = 93{,}845 / a_{\overline{348}|.5\%} = 569.64$$
$$\Delta = 1000 - 569.64 = 430.36$$

A

The answer is different by a few pennies, but same range

THIS PAGE WAS INTENTIONALLY LEFT BLANK

1994 Spring EA-1A   Added solution 2/23/99

15 There are two ways to work this problem. I'll start with the most logical one, which is also fairly visual and intuitive. The second solution is much less reproducible under exam conditions, and thus is much less desirable!

This first approach is called the "method of detached coefficients." In this method of solution, you set up a sparse matrix that is easily solvable. (Also, see problem 19)

The first step is writing down a general expression for the present value using all of the annuities. We'll be vague and use $\ddot{a}_{SG}$ for the annuity payable to both Smith and Green, $\ddot{a}_{BS}$ for the annuity to Brown and Smith, etc

$$PV = S\cdot\ddot{a}_S + B\cdot\ddot{a}_B + G\cdot\ddot{a}_G + X\cdot\ddot{a}_{SB} + Y\cdot\ddot{a}_{SG} + Z\cdot\ddot{a}_{BG} + W\cdot\ddot{a}_{SBG}$$

The matrix will allow you to solve for the value of each coefficient. Simply write an expression for the payment amounts given in the problem:

All 3 alive: $12{,}500 = S + B + G + X + Y + Z + W$
S/B only alive: $12{,}500 = S + B + X$
B/G only "alive": $9{,}000 = B + G + Z$
S/G only "alive": $12{,}500 = S + G + Y$
S only alive: $10{,}000 = S$
B only alive: $6{,}000 = B$
G only "alive": $3{,}000 = G$

Note that payments to Green are assumed to end at age 18, which is why "alive" is in quotes. Now you can calculate the values of each coefficient, working from the

Spring 1994 EA-1A

(15) bottom up to the top of the matrix. You already have values for S, B, and G. Now you can solve for the values of X, Y, and Z:

$$12{,}500 = S+G+Y \quad \therefore Y = 12{,}500-(10{,}000+3{,}000) = -500$$
$$9{,}000 = B+G+Z \quad \therefore Z = 9{,}000-(6{,}000+3{,}000) = 0$$
$$12{,}500 = S+B+X \quad \therefore X = 12{,}500-(10{,}000+6{,}000) = -3{,}500$$

Finally, you can solve for the value of W:

$$\begin{aligned} 12{,}500 &= (S+B+G)+(X+Y+Z)+W \\ &= (10{,}000+6{,}000+3{,}000)+(-3{,}500+0-500)+W \\ &= 19{,}000-4{,}000+W \\ W &= -2{,}500 \end{aligned}$$

The last step is calculation of the present value using the appropriate annuity values with the various coefficients you just calculated:

$$\begin{aligned} PV &= S\cdot\ddot{a}_S+B\cdot\ddot{a}_B+G\cdot\ddot{a}_G+X\cdot\ddot{a}_{SB}+Y\cdot\ddot{a}_{SG}+Z\cdot\ddot{a}_{BG}+W\cdot\ddot{a}_{SBG} \\ &= 10{,}000(\ddot{a}_{45})+6{,}000(\ddot{a}_{35})+3{,}000(\ddot{a}_{\overline{10|}})\\ &\quad -3{,}500(\ddot{a}_{45:35})-500(\ddot{a}_{45:\overline{10|}})-0-2{,}500(\ddot{a}_{45:35:\overline{10|}}) \\ &= 10{,}000(15)+6{,}000(20)+3{,}000(7)\\ &\quad -3{,}500(12)\ -500(6)\ -2{,}500(5) \\ &= 233{,}500 \end{aligned}$$

E

The advantages of this solution over that shown on the next 2 pages are as follows:
(i) use of a standard method that is reusable
(ii) did not take into account specifics of annuity definitions until the last step
(iii) directly handled the maximum payment amounts.

Spring 1994 EA-1A Solutions

(15) The maximum annual payment of $12,500 makes this a very tricky problem. The best way to work it is to set up a table, and to identify the amounts provided under various cases. Note that the annuity that pays Green, but not beyond age 18, could be considered quite like any other life annuity. The fact that it is a Temporary annuity does not alter the method of solution.

Annuity payable to	Who is alive? Smith and Brown only	Smith and Green only	Smith Brown and Green
Smith = 10,000	10,000	10,000	10,000
Brown = 6,000	6,000	—	6,000
Green = 3,000	—	3,000	3,000
Total payments	16,000	13,000	19,000
Payment > 12,500	3,500	500	6,500

Now you need to deduct the value of annuities payable to the various joint life statuses to reduce the payments to 12,500. For Smith:Brown you need a reduction of 3,500, and for Smith:Green you need a reduction of 500. After reflecting those in the third column, you can derive the reduction required when all three are alive.

(next page)

Spring 1994 EA-1A Solutions

(15) Continued

Annuity payable to	Who is alive? Smith and Brown only	Smith and Green only	Smith, Brown and Green
Smith = 10,000	10,000	10,000	10,000
Brown = 6,000	6,000	—	6,000
Green = 3,000	—	3,000	3,000
Total payment	16,000	13,000	19,000
Smith : Brown = -3,500	-3,500	—	-3,500
Smith : Green = - 500	—	-500	- 500
All three = -2,500	—	—	-2,500
Total payment	12,500	12,500	12,500

Now, you must calculate the value of the various annuities:

10,000

Smith $= 10{,}000\,\ddot{a}_{45} = 10{,}000(15) = 150{,}000$

Brown $= 6{,}000\,\ddot{a}_{35} = 6{,}000(20) = 120{,}000$

Green $= 3{,}000\,\ddot{a}_{10:\overline{81}} = 3{,}000(7) = 21{,}000$

Smith : Brown $= -3{,}500\,\ddot{a}_{45:35} = -3{,}500(12) = -42{,}000$

Smith : Green $= -500\,\ddot{a}_{45:10:\overline{81}} = -500(6) = -3{,}000$

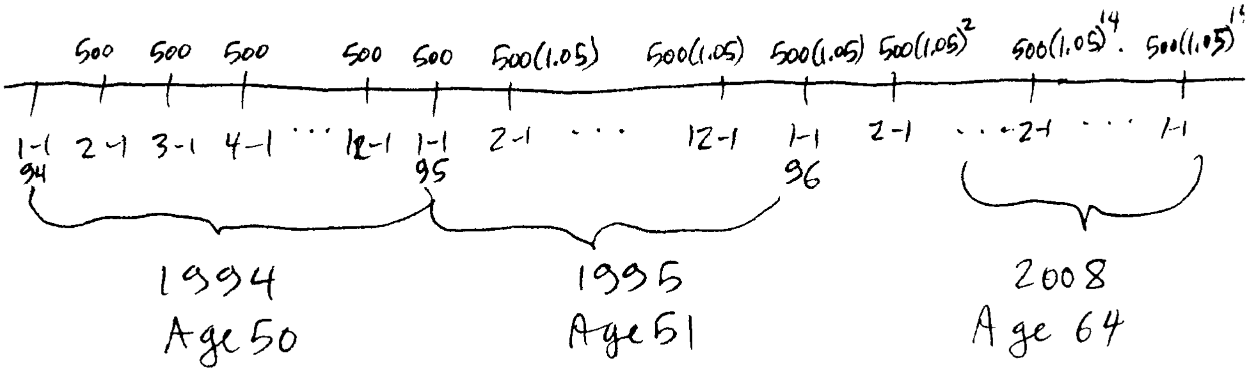
All three $= -2{,}500\,\ddot{a}_{45:35:10:\overline{81}} = -2{,}500(5) = -12{,}500$

233,500

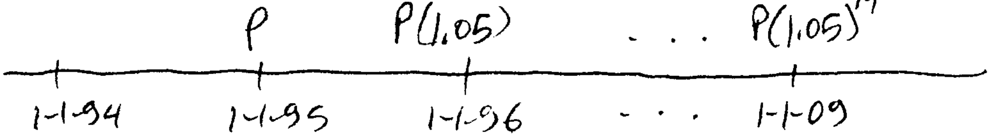
(E)

Spring 1994 EA-1A Solutions

16 The first step is to draw a time diagram of the series of deposits to Smith's account.



The easiest way to value this series is to replace the monthly payments with a single payment of P at 12/31, which would grow at the rate of 5% per year.



$(1.07)^{1/12} - 1 = 1.00565 \Rightarrow$ monthly interest of .565%

$P = 500\, s_{\overline{12}|.565\%}$

$= 6190.15$

At 1-1-94, the present value of the payments is

$$PV = \frac{1}{1.07}\left[P\left(1 + \frac{1.05}{1.07} + \dots + \left(\frac{1.05}{1.07}\right)^{14}\right)\right] \qquad \frac{1.07}{1.05} = 1.0190$$

$$= (6190.15/1.07)\ \ddot{a}_{\overline{15}|1.9\%}$$

Account value at age 65 $= (1.07)^{14}(6190.15)\ \ddot{a}_{\overline{15}|1.9\%}$

$= 210,497$

Monthly benefit $= \dfrac{210,497}{200} = 1052$ B

Solution corrected 2/23/99

Spring 1994 EA-1A Solutions

17 This problem tests your knowledge of some insurance identities. You don't know the issue age for the policy, but you are given the premium rate, and the net level reserve:

$$_tV_x = F(A_{x+t} - P_x \cdot \ddot{a}_{x+t}) \quad \text{at age } x+t$$

$$127.84 = F(A_{58} - .01737\,\ddot{a}_{58}) \quad \text{at age } 58$$

$$= F(1 - d\ddot{a}_{58} - .01737\,\ddot{a}_{58})$$

$$= F\left(1 - (.01737 + d)\frac{N_{58}}{D_{58}}\right)$$

$$F = \frac{127.84}{1 - \left(.01737 + \frac{.07}{1.07}\right)\left[\frac{1{,}809{,}565}{169{,}509}\right]}$$

$$= 1100.33$$

Next, set up the equation for the net level reserve at age 62:

$$_tV_x = F(A_{62} - .01737\,\ddot{a}_{62}) \quad \text{at age } 62$$

$$= 1100.33\left[1 - d\,\ddot{a}_{62} - .01737\,\ddot{a}_{62}\right]$$

$$= 1100.33\left[1 - \left(.01737 + \frac{.07}{1.07}\right)\left(\frac{1{,}206{,}064}{122{,}414}\right)\right]$$

$$= 202.81$$

E

Solution corrected 2-19-2000

Spring 1994 EA-1A Solutions

18 This problem is mostly a probability question, with a small bit of population theory. The number of members attaining age 55 is ... and the number attaining age 56 is ℓ_{56}. The calculation of the number of deaths is easy

Number who will die between age 55 and 57 is

$$d_{55}+d_{56} = \ell_{55}-\ell_{57} = \ell_{55}(1-{}_2p_{55}) = \ell_{55}(1-p_{55}\cdot p_{56})$$

Number who will die between age 56 and 57 is

$$d_{56} = \ell_{56}-\ell_{57} = \ell_{56}(1-p_{56}) = \ell_{55}\cdot p_{55}(1-p_{56})$$

Total number of deaths:

$$\ell_{55}(1-p_{55}\cdot p_{56}) + \ell_{55}\cdot p_{55}(1-p_{56})$$

$$= \ell_{55}[1 - p_{55}\cdot p_{56} + p_{55} - p_{55}\cdot p_{56}]$$
$$= \ell_{55}[1 + p_{55}(1-2\cdot p_{56})]$$

You know that $vp_x(\ddot{a}_{x+1}) = a_x$, so this can be used to derive the p_x values from e_x by setting $i=0$:

$$vp_x(1+a_{x+1}) = a_x \Rightarrow p_x(1+e_{x+1}) = e_x$$
$$p_x = e_x/(1+e_{x+1})$$

$p_{55} = e_{55}/(1+e_{56}) = 22.245/22.447 = .9910$

$p_{56} = e_{56}/(1+e_{57}) = 21.447/21.661 = .9901$

$$\ell_{55}[1+p_{55}(1-2\cdot p_{56})] = 100{,}000[1+.9910(1-2(.9901))] = 2862$$ D

Spring 1994 EA-1A Added solution 2/23/99

(19) There are two ways to work this problem. The most direct approach is writing down all of the possible combinations of annuities, which is fairly tedious and error prone.

There is an alternative method of solution that is fairly visual and intuitive, and therefore preferable. It is called the "method of detached coefficients." In this method of solution, you set up a sparse matrix which is easily solvable. (Also, see problem 15)

The first step is writing down a general expression for the present value which uses all of the annuities:

$$PV = X \cdot a_x + Y \cdot a_y + Z \cdot a_z + A \cdot a_{xy} + B \cdot a_{xz} + C \cdot a_{yz} + W \cdot a_{xyz} + P \cdot a_{\overline{\infty}|i}$$

The matrix will allow us to solve for the values of each coefficient. Simply write an expression for the payment amounts given in the problem:

3 alive:	$150 = X + Y + Z + A + B + C + W + P$	
2 alive:	$120 = X + Y + A + P$	(x, y only)
"	$120 = X + Z + B + P$	(x, z only)
"	$120 = Y + Z + C + P$	(y, z only)
1 alive:	$100 = X + P$	(x only)
"	$100 = Y + P$	(y only)
"	$100 = Z + P$	(z only)
0 alive	$50 = P$	all dead

Note that the perpetuity pays something regardless of who is alive or dead. You can now solve for the values of each coefficient, working from

Spring 1994 EA-1A Added solution 2/23/99

(19) the bottom up. First, you know that $P = 50$.

Now you can solve for values of X, Y and Z:

$100 = Z + P \quad \therefore Z = 100 - 50 = 50$

$100 = Y + P \quad \therefore Y = 50$

$100 = X + P \quad \therefore X = 50$

Now you can solve for values of A, B, and C.

$120 = Y + Z + C + P \therefore C = 120 - (Y + Z + P) = -30$

$120 = X + Z + B + P \quad \therefore B = -30$

$120 = X + Y + A + P \quad \therefore A = -30$

Finally, you can solve for the value of W:

$$\begin{aligned} 150 &= X + Y + Z + A + B + C + W + P \\ &= (50+50+50) + (-30-30-30) + W + 50 \\ &= 150 - 90 + W + 50 \\ &= 110 + W \\ W &= 40 \end{aligned}$$

The last step is calculation of the present value using all the coefficients and the annuity values:

$$\begin{aligned} PV &= Xa_x + Ya_y + Za_z + Aa_{xy} + Ba_{xz} + Ca_{yz} + Wa_{xyz} + Pa_{\overline{\infty}|} \\ &= 50(a_x + a_y + a_z) - 30(a_{xy} + a_{xz} + a_{yz}) + 40a_{xyz} + 50/.07 \\ &= 50(38) - 30(35) + 40(11) + 50/.07 \\ &= 1{,}900 - 1{,}050 + 440 + 714 \\ &= 2{,}004 \end{aligned}$$

C

On the next page is the alternative solution, which requires writing down lots (and lots) of joint life annuities (very carefully)

Spring 1994 EA-1A Solutions

(19) The trick to this problem is correctly writing down all the reversionary annuities for all the cases. In addition, it is a bit complicated due to the necessity to handle a joint and last survivor annuity for three lives.

$$a_{\overline{xy}} = a_x + a_y - a_{xy}$$
$$a_{\overline{xyz}} = a_{\overline{x:\overline{yz}}}$$
$$= a_x + a_{\overline{yz}} - a_{x:\overline{yz}}$$
$$= a_y + (a_y + a_z - a_{yz}) - (a_{xy} + a_{xz} - a_{xyz})$$
$$= (a_x + a_y + a_z) - (a_{xy} + a_{xz} + a_{yz}) + a_{xyz}$$

$$PV = 150\,a_{xyz} + 120[(a_{xy} - a_{xyz}) + (a_{xz} - a_{xyz}) + (a_{yz} - a_{xyz})]$$
$$+ 100[(a_x - a_{x:\overline{yz}}) + (a_y - a_{y:\overline{xz}}) + (a_z - a_{z:\overline{xy}})]$$
$$+ 50[1/i - a_{\overline{xyz}}]$$

$$= 150\,a_{xyz}$$
$$+120\,[a_{xy} + a_{xz} + a_{yz} - 3(a_{xyz})]$$
$$+100\,[a_x - (a_{xy} + a_{xz} - a_{xyz})]$$
$$+100\,[a_y - (a_{y\cancel{x}} + a_{yz} - a_{xyz})]$$
$$+100\,[a_z - (a_{zx} + a_{zy} - a_{xyz})]$$
$$+ 50\left[\frac{1}{.07} - (a_x + a_y + a_z - (a_{xy} + a_{xz} + a_{yz}) + a_{xyz})\right]$$

$$= 150\,(11)$$
$$+120\,(35 - 3\cdot 11)$$
$$+100\,(38 - 2\cdot 35 + 3\cdot 11)$$
$$+ 50\left(\frac{1}{.07} - 38 + 35 - 11\right)$$

$$= 1650 + 120(2) + 100(1) + 50(.2857)$$
$$= 2004$$

(C)

Spring 1994 EA-1A Solutions

20 There are two ways to work this problem, depending on whether you believe what you are told in the problem. I don't believe that you'll end up with a final payment of 1500 after contributing an extra 10,000 towards the outstanding loan balance. The shorter solution assumes that all payments equal 1500:

1-1-94 initial loan balance $= 1500\, a_{\overline{360}|.5\%} = 250{,}187$

1-1-97 O/S loan balance without 10,000 add'l $= 1500\, a_{\overline{324}|.5\%} = 240{,}390.26$

Revised loan balance $= 240{,}390 - 10{,}000 = 230{,}390.26$

Number of remaining payments of 1500/mo $\Rightarrow \dfrac{230{,}390}{1500} = a_{\overline{n}|.5\%} \Rightarrow n = 293$

Δ Interest $= \Delta$ principal over life of loan

original loan payments $= 360(1500) = 540{,}000$

revised loan payments $= (36+293)1500 + 10{,}000 = 503{,}500$

$\Delta = 36{,}500$

C

The more exact approach is to recognize that the 293rd payment is not 1500:

$1500\, a_{\overline{293}|} = 230{,}423$

$1500\, a_{\overline{292}|} = 230{,}075.20 \quad \Delta = 315.06$ (versus 230,390.26)

last payment $= 315.06\,(1.005)^{292} = 1351.69$

Revised loan payments $= (36+292)1500 + 1352 + 10{,}000 = 503{,}352$

$\Delta = 36{,}648 = (540{,}000 - 503{,}352)$

Spring 1994 EA-1A Solutions

21 This is one of the few questions that has been asked for a triple decrement table. In lieu of the approximation formulas, which I don't know for other than a double decrement table, you can use Bowers' formula:

$$p_x'^{(b)} \doteq \left[p_x^{(T)}\right]^{q_x^{(b)}/q_x^{(T)}}$$

$$= (1-.02-.07-.06)^{.07/.15}$$

$$= .9270$$

$$q_x'^{(b)} = .073$$

C

Based on other exam solutions, I have seen this approximation for a triple decrement rate in terms of the probabilities:

$$q_x'^{(b)} \doteq \frac{q_x^{(b)}}{1-\frac{1}{2}q_x^{(a)}-\frac{1}{2}q_x^{(c)}}$$

This seems reasonable based on the assumption of uniform distribution of decrements.

$$q_x'^{(b)} \doteq \frac{.07}{1-.5(.02)-.5(.06)}$$

$$= .0729$$

Spring 1994 EA-1A Solutions

22 You should draw a time line diagram for the series of increasing monthly payments

10	20	30	...
1-94	2-94	3-94	...

Let m be the monthly interest rate. You can also write the series of payments as multiple layers of level perpetuities, assuming you don't remember the formula

		10	...
	10	10	...
10	10	10	...
1-94	2-94	3-94	...

let $P = 10 + 10v + 10v^2 + \ldots = 10\,\ddot{a}_{\overline{\infty}|m} = 10/d$

$320{,}000 = P + vP + v^2P + \ldots = P/d = 10/d^2$

$\therefore d^2 = \dfrac{1}{32{,}000} \Rightarrow \dfrac{m}{1+m} = \dfrac{1}{178.89}$

$178.89m = 1 + m$

$m = \dfrac{1}{177.89} = .005622$

Annual interest rate $x = (1+m)^{12}$

$= (1.005622)^{12}$

$= 1.0696$

D

Spring 1994 EA-1A Solutions

23 The first step is to draw a time line diagram for the increasing death benefits:

	25,000	50,000	75,000	100,000	100,000	. . .
100,000	100,000	100,000	100,000	100,000	100,000	. . .
1-94	1-95	1-96	1-97	1-98	1-99	. . .
65	66	67	68	69	70	. . .

You can express this in terms of the commutation functions R_x and M_x. The main trick is that you'll have to subtract an increasing insurance starting at age 70 to produce the level 200,000 death benefit:

$$P \cdot \frac{N_{65}}{D_{65}} = PVB = \frac{100{,}000(M_{65}) + 25{,}000(R_{66}) - 25{,}000(R_{70})}{D_{65}}$$

Now you need to use some identities to obtain values for M_{65} and N_{65}. The D_{65} values are not needed to calculate the premium (P):

$$M_x = D_x - d \cdot N_x \Rightarrow R_x = N_x - d \cdot S_x \therefore N_{65} = R_{65} + d \cdot S_{65}$$

$$N_{65} = 9{,}523{,}140 + (.06/1.06)(139{,}445{,}171)$$
$$= 17{,}416{,}263$$

$$M_{65} = R_{65} - R_{66} = 9{,}523{,}140 - 8{,}732{,}440$$
$$= 790{,}700$$

$$P = \frac{100{,}000(790{,}700) + 25{,}000(8{,}732{,}440 - 5{,}952{,}189)}{17{,}416{,}263}$$
$$= 8531$$

D

Spring 1994 EA-1A Solutions

24 The first thing to do in this problem is to look at the present value of the normal retirement benefit payable at age 65:

$$PVB@65 = 1000 + v\,p_{65}(1000)(1.03) + v^2\,{}_2p_{65}(1000)(1.03)^2 + \ldots$$
$$= 1000\left[1 + p_{65}\left(\frac{1.03}{1.07}\right) + {}_2p_{65}\left(\frac{1.03}{1.07}\right)^2 + \ldots\right]$$
$$= 1000\,\ddot{a}_{65}^{j} \qquad \text{where } 1+j = \frac{1.07}{1.03} \Rightarrow j = \frac{.04}{1.03}$$

When Smith retires at age 60, you know that the present value of the early retirement benefit is actuarially equivalent to the normal retirement benefit:

$$PVB@60 = ERB\left[1 + p_{60}\left(\frac{1.03}{1.07}\right) + {}_2p_{60}\left(\frac{1.03}{1.07}\right)^2 + \ldots\right]$$
$$= ERB\,\ddot{a}_{60}^{j}$$

$$ERB\,\ddot{a}_{60}^{j} = \frac{D_{65}^{7\%}}{D_{60}^{7\%}}(1000)\,\ddot{a}_{65}^{j}$$

$$\therefore ERB = \frac{D_{65}^{7\%}}{D_{60}^{7\%}}(1000)\,\frac{\ddot{a}_{65}^{j}}{\ddot{a}_{60}^{j}}$$
$$= \frac{94{,}414}{144{,}405}(1000)\left(\frac{7{,}339{,}325/644{,}844}{11{,}170{,}538/850{,}777}\right)$$
$$= .6538\,(1000)(.8668)$$
$$= 567$$

B

Spring 1994 EA-1A Solutions

25 The trick to this problem is that the amount of interest paid in each year must equal the outstanding loan balance at the preceding January 1st, multiplied by the rate of interest.

1-1-93 Loan balance $= \dfrac{14{,}700}{.07} = 210{,}000$

1-1-94 O/S loan $= 210{,}000\left(\dfrac{a_{\overline{2}|.07}}{a_{\overline{3}|.07}}\right)$

$= 144{,}679$

1994 interest rate $= \dfrac{8{,}681}{144{,}679} = 6\%$

1-1-95 O/S loan $= 144{,}679\left(\dfrac{a_{\overline{1}|.06}}{a_{\overline{2}|.06}}\right)$

$= 74{,}446$

Final payment should be $(1+i)\,74{,}446 = 81{,}600$

Interest paid at 12-31-85 $= 81{,}600 - 74{,}446$

$= 7{,}153$

D