



Software Polish

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SPRING 1997 EA-1A EXAM SOLUTIONS

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Revision History:

02/19/00	Enhanced problem 04	added clearer method of solution
	Corrected problem 07	corrected typo – “Month” instead of “Year”
	Enhanced problem 20	added clearer method of solution
	Corrected problem 21	Corrected answer letter

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1. You must calculate the time weighted return first, then use that result to solve for the date X. Set up a time diagram to calculate the market values after each cash flow, which are needed to calculate the time weighted return:

Date	1/1	7/1	X	12/31
Cash flow	-0-	-50,000	100,000	-0-
MV. before	1,000,000	1,030,000	1,025,000	1,150,000
MV. after	1,000,000	980,000	1,125,000	1,150,000

Now calculate ratios at the point of each cash flow:

$$1+i = \left(\frac{1,030,000}{1,000,000} \right) \left(\frac{1,025,000}{980,000} \right) \left(\frac{1,150,000}{1,125,000} \right)$$

$$= (1.0300)(1.0459)(1.0222) = 1.1012$$

The dollar weighted return is also equal to 10.12%. Set up a formula which equates interest on the market value plus each cash flow to the ending market value:

$$1,000,000 \left(1 + \left(\frac{12}{12} \right) i \right) - 50,000 \left(1 + \frac{6}{12} (i) \right) + 100,000 (1 + Y i) = 1,150,000$$

This assumes that date X corresponds to a fraction Y of a whole year for exposure to the force of interest.

$$i = \frac{100,000}{975,000 + 100,000 Y} = .1012 \quad (\text{after simplifying})$$

$$Y = \frac{1,295}{10,124} = .12793 \text{ year}$$

If X = 12/1/97, then Y would be .0833

If X = 11/1/97, then Y would be .1667

Y corresponds to date between 11/1 and 12/31

(E)

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2. you need to know how to set up a loan amortization schedule for this solution:

Year	Payment	Interest	Principal	Outstanding Loan Amount
0				$100 a_{\overline{n} i}$
1	100	$100(i a_{\overline{n} i})$	$100(1 - i a_{\overline{n} i})$	$100 a_{\overline{n-1} i}$
2	100	$100(1 - v^{n-1})$	$100(v^{n-1})$	$100 a_{\overline{n-2} i}$
\vdots	\vdots	\vdots	\vdots	\vdots
$n-1$	100	$100(1 - v^2)$	$100v^2$	$100 a_{\overline{1} i}$
n	100	$100(1 - v)$	$100v$	-0-

The amount of interest in the last 12 payments is
 $100((1-v) + (1-v^2) + \dots + (1-v^{12})) = 100(12 - a_{\overline{12}|i}) = 109.20$

Now you can solve for the interest rate for the loan

$$1200 - 100 a_{\overline{24}|i} = 109.20 \Rightarrow a_{\overline{24}|i} = 10.9080 \Rightarrow i = 1.4993\%$$

The amount of interest paid in the middle 12 payments is
 $100((1-v^{13}) + (1-v^{14}) + \dots + (1-v^{24}))$

$$= 100(12 - a_{\overline{24}|i} + a_{\overline{12}|i})$$

$$= 100(12 - a_{\overline{24}|1.4993\%} + a_{\overline{12}|1.4993\%})$$

$$= 100(12 - 20.0321 + 10.9080)$$

$$= 287.60$$

(B)

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3. You must determine the original loan payment and the total loan interest based on the original provisions. Then you can redetermine these items based on the renegotiated loan provisions.

Nominal rate of 10% per annum compounded monthly is equivalent to .8333% per month
monthly payment = $\frac{100,000}{a_{\overline{360}|.8333\%}} = 877.57$

$$\begin{aligned}\text{Total loan payments} &= 360 (877.57) = 315,925 \\ \text{Total interest paid} &= 315,925 - 100,000 = 215,925\end{aligned}$$

At 1-1-97, five years after the original date of the loan, 60 payments have been made.
1-1-97 O/S

$$\text{Loan balance} = 877.57 (a_{\overline{300}|.8333\%}) = 96,574$$

New nominal rate of 9% per annum compounded monthly is equivalent to .75% per month
monthly payment = $\frac{96,574}{a_{\overline{180}|.75\%}} = 979.52$

$$\begin{aligned}\text{Revised total loan payments} &= 60 (877.57) + 180 (979.52) \\ &= 228,967\end{aligned}$$

$$\begin{aligned}\text{Total interest paid} &= 228,967 - 100,000 = 128,967 \\ \text{A Interest paid} &= 128,967 - 215,925 = (86,958)\end{aligned}$$

You have to be careful to reflect the 60 payments already made in calculating the total loan payment. It produces the same result if you simply look at the change in total loan payments. ©

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- 4) The first step is calculation of the original loan payment $P = \frac{200,000}{a_{\overline{360}|.75\%}} = 1609.25$

Instead of assuming the additional loan payment is 1609.25 at 12/31, it is easier to work the problem assuming payments of Q at the end of each month. This is not quite accurate for the final year of payment, so we have to allow for the final year adjustment later.

$$Q = \frac{P}{5\overline{12}|.75\%} = 128.66$$

If we assume payments of $P+Q$ are made each month, we know how many loan payments will be necessary:

$$200,000 = (P+Q) a_{\overline{n}|.75\%} \quad a_{\overline{n}|.75\%} = \frac{200,000}{1609.25 + 128.66}$$

$$a_{\overline{n}|.75\%} = 115.08 \Rightarrow n = 267$$

The first guess for total number of payments is 360-267, or 93 payments less. Now we need to redetermine the actual number, based on annual payments at 12/31 of 1609.25, instead of the monthly payments of 128.66:

$$200,000 = 1609.25 (a_{\overline{n}|.75\%}) + 1609.25 (a_{\overline{22}|9.38\%})$$

I used 22 years for the annual payments based on the prior result of 267 monthly payments ($267/12 = 22.25$).

When you solve for n this time, it is still equal to 267!

The answer is that there are 93 fewer payments (D)

Here is why 267 is still the same:

$$(1609.25) a_{\overline{267}|.75\%} + (128.66) a_{\overline{267}|.75\%} = 200,205$$

$$(1609.25) a_{\overline{267}|.75\%} + (1609.25) a_{\overline{22}|9.38\%} = 200,152$$

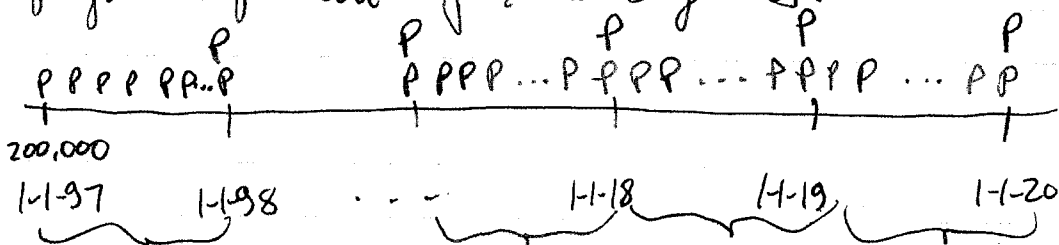
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(4) Continued - Alternate (longer) solution

This is a potentially tricky problem. First you must calculate the value of P based on the original 360 payments:

$$P = \frac{200,000}{a_{360|7.5\%}} = 1609.25$$

Now we'll pay 1609.25 each 12/31 in addition to the scheduled monthly payment at each 12/31. Based on the answer ranges, this will shorten the payment period by 7 to 8 years:



Year 1 Year 21 Year 22 Year 23

I decided to use trial and error, and to test the answers. First I'll assume there are 84 less payments, which means the last payments are made at 12/31/19:

$$PV \text{ of payments} = 1609.25 (a_{276|7.5\%} + a_{23|9.38\%})$$

$$202,255 = 187,281 + 14,974$$

The 9.38% rate is calculated as $(1.0075)^{12} - 1 = .0938$.
The trick is the separate handling of the 12/31 annual payments.

Since the value of these payments exceeds 200,000, we can get by with fewer payments. The next "whole year" in the answer ranges is 96 payments, which corresponds to the E answer range.

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(4) Assume the last payments occur at 12/31/18:

$$\begin{aligned} \text{PV of payments} &= 1609.25 (a_{\overline{26}|7.75\%} + a_{\overline{22}|9.38\%}) \\ 99,491 &= 184,721 + 14,770 \end{aligned}$$

Since this total is less than 200,000, we need to make additional payments in the year 2019. We previously determined that it was NOT necessary to make all payments through 12/31/19, since that total exceeded 200,000.

Continuing with the trial and error approach, you should calculate the result based on 91 fewer payments: $360 - 91 = 269$

$$\begin{aligned} \text{PV} &= 1609.25 (a_{\overline{269}|7.75\%} + a_{\overline{22}|9.38\%}) \\ 200,586 &= 185,816 + 14,770 \end{aligned}$$

Since this also creates an overpayment, the final answer is a reduction in payments between 91 and 95

(D)

You can solve for the value of n that produces a PV of 200,000, but it was necessary to previously determine which year the last payment is made:

$$\begin{aligned} 200,000 &= 1609.25 (a_{\overline{n}|7.75\%} + a_{\overline{22}|9.38\%}) \\ a_{\overline{n}|7.75\%} &= (200,000 - 14,770) / 1609.25 = 115.103 \\ \therefore n &= 267 \end{aligned}$$

Plug 267 into formula to verify results:

$$200,153 = 1609.25 (a_{\overline{267}|7.75\%} + a_{\overline{22}|9.38\%})$$

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- 5 This problem tests your ability to handle nominal rates of discount and interest. It is intentionally confusing too!

	Fund A	Fund B
1 st Ten years	$i^{(4)} = .06$	$d^{(12)} = .09$
$1+i$	$\left(1 + \frac{.06}{4}\right)^4 = 1.0614$	$\left(\frac{1}{1 - \frac{.09}{12}}\right)^{12} = 1.0945$

Value after 10 yrs	$W(1.0614)^{10}$	$X(1.0945)^{10}$
--------------------	------------------	------------------

2 nd Ten years	$d^{(4)} = .09$	$i^{(12)} = .12$
$1+i$	$\left(\frac{1}{1 - \frac{.09}{4}}\right)^4 = 1.0953$	$\left(1 + \frac{.12}{12}\right)^{12} = 1.1268$

Value after 20 yrs	$W(1.0614)^{10}(1.0953)^{10}$	$X(1.0945)^{10}(1.1268)^{10}$
	$= Y$	$= Z$

$$Y + Z = 57,186 = W(1.0614)^{10}(1.0953)^{10} + X(1.0945)^{10}(1.1268)^{10}$$

$$= W(1.8140)(2.4850) + X(2.4680)(3.3004)$$

$$57,186 = 4.5079W + 8.1452X$$

$$X + W = 10,000 \Rightarrow X = 10,000 - W$$

Substitute this in the prior equation:

$$57,186 = 4.5079W + 8.1452(10,000 - W)$$

$$3.6374W = 24,266$$

$$\therefore W = 6,671$$

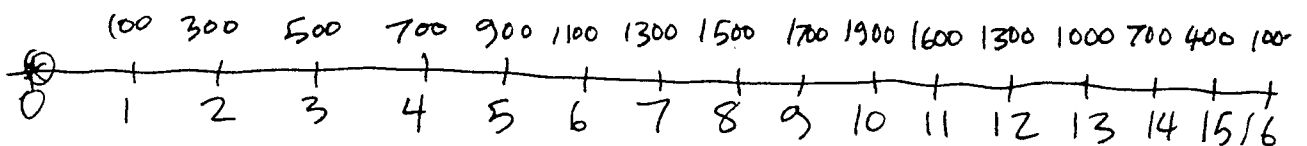
$$Y = W(1.8140)(2.4850)$$

$$= 30,074$$



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6. When you have an unusual series of payments, you can either attempt to set up annuities to represent the payments, or simply write an expression for the present value based on "first principles". I'll do the latter:



$$\begin{aligned}
 PV = X &= 100v + 300v^2 + 500v^3 + \dots + 1700v^9 + 1900v^{10} + 1600v^{11} + 1300v^{12} + \dots + 100v^{16} \\
 v \cdot X &= 100v^2 + 300v^3 + \dots + 1500v^9 + 1700v^{10} + 1900v^{11} + 1600v^{12} + \dots + 400v^{16} + 100v^{17} \\
 (1-v)X &= 100v + 200v^2 + 200v^3 + \dots + 200v^9 + 200v^{10} - 300v^{11} - 300v^{12} - \dots - 300v^{16} - 100v^{17}
 \end{aligned}$$

9 payments @ 200
6 payments @ 300

$$\begin{aligned}
 (1-v)X &= 100v + 200v(a_{\overline{9}|1.07}) - 300v^{10}(a_{\overline{6}|1.07}) - 100v^{17} \\
 X &= \frac{100}{1.07} + \frac{200 a_{\overline{9}|1.07}}{1.07} - \frac{300 a_{\overline{6}|1.07}}{(1.07)^{10}} - \frac{100}{(1.07)^{17}} \\
 &= \frac{93.46 + 186.92(6.5152) - 152.50(4.7665) - 31.66}{1 - (1.07)^{-1}} \\
 &= 552.68 / 1.0654 \\
 &= 8,448
 \end{aligned}$$

(C)

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- 7 The first step is determining the equivalent monthly interest rates for the loan
- 1st 15 years 2nd 15 years

$$(1+i) \text{ monthly} \quad \frac{1.07}{(1.07)^{1/2}} = 1.57\% \quad \frac{1.11}{(1.11)^{1/2}} = 1.87\%$$

Now you can determine the level payment to pay off the loan in 180 payments:

$$1,000,000 = P [a_{\overline{180}|1.57\%} + (1.07)^{-15} a_{\overline{180}|1.87\%}]$$

$$P = \frac{1,000,000}{112.76 + .3624(90.56)}$$

$$= 6869.01$$

You should set up the amortization schedule to find the interest in the 204th repayment. The schedule is much messier for the first 15 years than for the second 15 years:

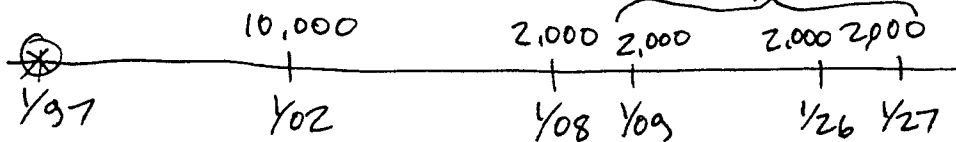
Month	Payment	Interest	Principal	O/S loan
0	—	—	—	1,000,000 = $P(a_{\overline{180} 1.57\%} + (1.0057)^{-15} a_{\overline{180} 1.87\%})$
1	P	.0057(1,000,000)	P - 5654	$P(a_{\overline{179} 1.57\%} + (1.0057)^{-179} a_{\overline{180} 1.87\%})$
⋮	⋮	⋮	⋮	⋮
180	P			$P(a_{\overline{180} 1.87\%})$
181	P	.0087P		$P(a_{\overline{179} 1.87\%})$
⋮	⋮	⋮	⋮	⋮
204	P	.0087P		$P(a_{\overline{156} 1.87\%})$
		$= .0087(6869.01)(85.26)$		
		$= 5,115$		(A)

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8. Modified duration equals regular duration over $(1+i)$. Regular duration is a weighted average of the year of payment, where the weight is the present value of the payment:

$$\text{Modified duration} = \frac{\bar{d}}{1+i} \quad \bar{d} = \frac{\sum_{t=1}^n t v^t R_t}{\sum_{t=1}^n v^t R_t}$$

First you need to draw the time diagram of payments, then write out the formula for modified duration



$$\frac{\bar{d}}{1+i} = \frac{5v^5(10,000) + 11v^{11}(2,000) + 12v^{12}(2,000) + \dots + 30v^{30}(2,000)}{[v^5(10,000) + v^{11}(2,000) + v^{12}(2,000) + \dots + v^{30}(2,000)]1.08}$$

$$= \frac{5v^5(10,000) + 2,000v^{10}(11v + 12v^2 + \dots + 30v^{20})}{1.08(v^5(10,000) + 2,000v^{10}(v + v^2 + \dots + v^{20}))}$$

$$= \frac{5v^5(10,000) + 2,000v^{10}(10v^1 + \dots + 10v^{20} + v^1 + 2v^2 + \dots + 20v^{20})}{1.08[v^5(10,000) + 2,000v^{10}(v^1 + \dots + v^{20})]}$$

$$= \frac{50,000v^5 + 2,000v^{10}(10a_{\overline{20}|1.08} + (1a)_{\overline{20}|1.08})}{1.08[10,000v^5 + 2,000v^{10}(a_{\overline{20}|1.08})]}$$

$$= \frac{50,000v^5 + 2,000v^{10}(10a_{\overline{20}|1.08} + \frac{\ddot{a}_{\overline{20}|1.08} - 20v^{20}}{.08})}{1.08[10,000v^5 + 2,000v^{10}(a_{\overline{20}|1.08})]}$$

$$11.534 = \frac{34,029.16 + 926.39(98.18 + 78.91)}{1.08(6805.83 + 9095.40)} = \frac{198,082.49}{1.08(15,901.24)}$$

Ⓒ

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9. The formula for amortized value is the same as the price formula for a bond. The interest rate used to discount the coupons and the redemption value is the purchaser's yield rate:

6/30/97 amortized

$$\text{value before coupon} = 35.00 + 1000v^n + 35a_{\overline{n}|j} = 939.33$$

12/31/97 amortized

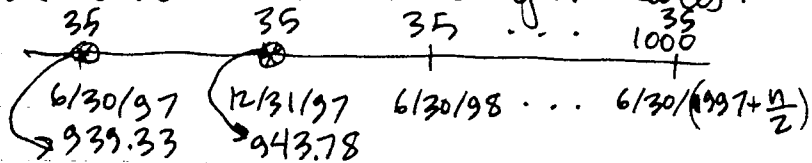
$$\text{value before coupon} = 35.00 + 1000v^{n+1} + 35a_{\overline{n+1}|j} = 943.78$$

The coupon of 35.00 corresponds to the 7% per year coupon rate, paid semiannually: $.035(1000) = 35.00$.

The bond will be redeemed n periods after 6/30/97, where each period is six months long.

I've used a rate j in the formulas above, since that is the semi-annual yield rate, not the annual yield rate.

When you look at a time diagram of the future payments received by the bond holder, there is a useful relationship between the amortized value at two successive coupon dates:



$$(939.33 - 35.00)(1+j) = 943.78$$

$$1+j = 943.78 / 904.33$$

$$= 1.0436$$

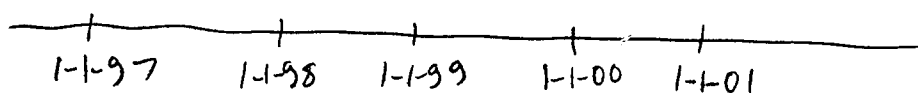
$$\text{Annual yield } 1+i = (1+j)^2 = (1.0436)^2 = 1.0891$$

(E)

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10. To solve problems with varying perpetuities, you may be able to apply pre-memorized formulas. Alternatively, you simply write out a time diagram and derive the present value based on "first principles":

$$\begin{array}{ccccc} (1)(2)/2 & 2(3)/2 & 3(4)/2 & 4(5)/2 & 5(6)/2 \\ = 1 & = 3 & = 6 & = 10 & = 15 \end{array}$$



The series of payments is the sum of the first n integers, which is relatively unique!

$$\begin{aligned} P &= 1 + 3v + 6v^2 + 10v^3 + 15v^4 + \dots \\ vP &= v + 3v^2 + 6v^3 + 10v^4 + \dots \\ P(1-v) &= 1 + 2v + 3v^2 + 4v^3 + 5v^4 + \dots \end{aligned}$$

At this point, you could apply the formula for an increasing perpetuity. But even if you don't know the formula, you can derive the value:

$$vP(1-v) = v + 2v^2 + 3v^3 + 4v^4 + \dots$$

$$P(1-v)^2 = 1 + v + v^2 + v^3 + v^4 + \dots$$

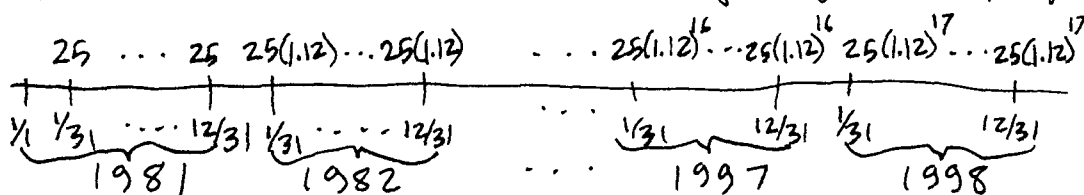
$$\lim_{n \rightarrow \infty} \frac{1-v^{n+1}}{1-v} = \frac{1}{1-v} = \lim_{n \rightarrow \infty} \frac{1-v^n}{d}$$

$$P = \frac{1}{(1-v)^3} = \frac{1}{\left(1 - \frac{1}{1.25}\right)^3} = \frac{1}{(.2)^3} = 125$$

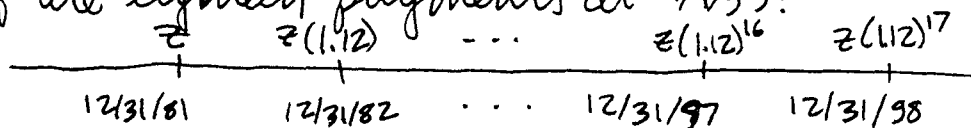
(D)

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11. You need to draw a time diagram for the payments



You can solve the problem by calculating this as the future value of twelve identical series of eighteen payments. An alternative approach is to replace the monthly payments with an annual payment, then determine the value of the eighteen payments at 1/1/99:



Based on the 12% per annum rate compounded monthly, the effective annual rate is $(1.01)^{12} - 1 = 12.68\%$. The annual payment Z is equivalent to the accumulated value of twelve payments of 25.00 at the 1% per month interest rate:

$$Z = 25 s_{\overline{12}|0.01} = 317.06$$

$$\text{1/1/99 accumulated value} = 317.06(1.1268)^{17} + 317.06(1.12)(1.1268)^{16} + \dots + 317.06(1.12)^{17}(1.1268)^0$$

Now factor out the first term, and you can evaluate the sum:

$$= 317.06(1.1268)^{17} \left[1 + \frac{1.1200}{1.1268} + \dots + \left(\frac{1.1200}{1.1268} \right)^{17} \right]$$

$$= 317.06(1.1268)^{17} \ddot{a}_{\overline{18}|j} \quad \text{where } 1+j = 1.1268/1.1200 = 1.0061$$

$$= 317.06(7.6131)(17.1026)$$

$$= 41,282.55$$

(D)

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12. This problem hinges on whether you can interpret correctly the final statement about the reserve at the end of the 19th year.

$${}_{19}V_x = PV(\text{Future Benefits}) - PV(\text{Future Premiums})$$
$${}_{19}V_{45} = 10,000 \left(\frac{M_{64} - M_{65} + D_{65}}{D_{64}} \right) - 350$$

There is only one premium left to be paid at age 64 in this reserve calculation. You are told that this reserve will provide 10,000 of term insurance, plus a pure endowment (X):

$${}_{19}V_{45} = 10,000 \left(\frac{M_{64} - M_{65}}{D_{64}} \right) + X \frac{D_{65}}{D_{64}}$$

Now you can equate these values, plug in D_{64} and D_{65} , and calculate the value of X

$$10,000 \left(\frac{M_{64} - M_{65}}{D_{64}} \right) + 10,000 \frac{D_{65}}{D_{64}} - 350 = 10,000 \left(\frac{M_{64} - M_{65}}{D_{64}} \right) + X \frac{D_{65}}{D_{64}}$$

$$(10,000 - X) \frac{D_{65}}{D_{64}} = 350$$

$$10,000 - \frac{D_{64}}{D_{65}} (350) = X$$

$$X = 10,000 - 350 \left(\frac{12,196}{11,240} \right)$$

$$= 9620.23$$

(B)

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13. This is a potentially confusing joint and last survivor annuity problem. The confusion is in figuring out the correct handling of the 11 year certain terms for the joint life annuity.

Since Smith and Brown's payments cease at age eighteen, you should consider the annuity payable to either one as a joint annuity with a certain period:

Smith $a_{10:\overline{8}|}$
Brown $a_{14:\overline{4}|}$

In general, for an annuity payable as long as one of two lives x and y is alive is

$$a_{\overline{xy}} = a_x + a_y - a_{xy}$$

Now replace x with Smith, and y with Brown

$$a_{\overline{xy}} = a_{\overline{10:\overline{8}| : 14:\overline{4}|}} = a_{10:\overline{8}|} + a_{14:\overline{4}|} - a_{10:\overline{8}|:14:\overline{4}|}$$

The point of the problem is that the final joint annuity can be written as $a_{10:\overline{14:\overline{4}|}}$. The reason is that no payments will be made under this annuity beyond the 4 years, since Brown will be over age 18. As far as the three terms for the joint and last survivor, the only payments made beyond 4 years from now will be to Smith, and are included in the $a_{10:\overline{8}|}$ term.

Net single premium

$$\begin{aligned} &= (a_{10:\overline{8}|} + a_{14:\overline{4}|} - a_{10:\overline{14:\overline{4}|}}) 1000 \\ &= (5.78 + 3.22 - 3.12) 1000 \\ &= 5880 \end{aligned}$$

NOTE: If you incorrectly use $a_{10:\overline{14:\overline{8}|}}$, your answer will be 3470, which is outside the implied range of 5500-5600. (D)

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14. There are two methods to use to estimate the multiple decrement probabilities at ages 40 and 41. The problem asks for $q_{40}^{(w)} + p_{40}^{(r)} q_{41}^{(w)}$.

Jordan's life contingencies gives several formulas to derive the single decrement rates based on multiple decrement probabilities (and vice versa).

Formula 14.31b: $q_x^{(1)} = \frac{q_x^{(1)}}{1 - \frac{1}{2} q_x^{(2)}}$

This formula is based on a double decrement table, and uniform distribution of decrements. You can use this formula to calculate the value of $q_{41}^{(w)}$ based on $q_{41}^{(d)}$ (indirectly).

$$p_{40}^{(r)} = \frac{93,674}{100,000} = .93674 \quad q_{40}^{(r)} = 1 - .93674 = .06326$$

$$p_{41}^{(r)} = \frac{87,867}{93,674} = .93801 \quad q_{41}^{(r)} = 1 - .93801 = .06199$$

$$q_{41}^{(d)} = \frac{q_{41}^{(d)}}{1 - \frac{1}{2} q_{41}^{(w)}} = \frac{q_{41}^{(d)}}{1 - \frac{1}{2} (q_{41}^{(r)} - q_{41}^{(d)})}$$

$$.0024 = \frac{q_{41}^{(d)}}{1 - .5(.06199 - q_{41}^{(d)})}$$

$$= \frac{q_{41}^{(d)}}{1 - .030995 + .5 q_{41}^{(d)}}$$

$$.0024 (.969005 + .5 q_{41}^{(d)}) = q_{41}^{(d)}$$

$$q_{41}^{(d)} = .002326 / .9988 = .002328$$

$$q_{41}^{(w)} = q_{41}^{(r)} - q_{41}^{(d)}$$

$$= .06199 - .002328 = .059662$$

$$q_{40}^{(w)} = q_{40}^{(r)} - q_{41}^{(d)}$$

$$= .06326 - .00213 = .06113$$

$$q_{40}^{(w)} + p_{40}^{(r)} q_{41}^{(w)} = .06113 + .93674 (.059662)$$

$$= .11702$$

(C)

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- (14). The ~~second~~ formula in Jordan can also be used to work this problem. This is a more general formula, which can be used for any number of decrements.

Formula 14.35 $p_x^{(1)} \doteq [p_x^{(T)}]^{\frac{q_x^{(d)}}{q_x^{(T)}}}$

$$p_{40}^{(T)} = \frac{93,674}{100,000} = .93674$$

$$q_{40}^{(T)} = 1 - .93674 = .06326$$

$$p_{41}^{(T)} = \frac{87,867}{93,674} = .93801$$

$$q_{41}^{(T)} = 1 - .93801 = .06199$$

$$1 - q_{41}^{(d)} \doteq [p_{41}^{(T)}]^{\frac{q_{41}^{(d)}}{q_{41}^{(T)}}}$$

$$1 - .0024 \doteq (.93801)^{\frac{q_{41}^{(d)}}{.06199}}$$

$$\begin{aligned} \log(.9976) &= \frac{q_{41}^{(d)}}{.06199} \left[\log(.93801) \right] \\ q_{41}^{(d)} &= .06199 \left(\frac{\ln(.9976)}{\ln(.93801)} \right) \\ &= .06199 \left(\frac{-.002403}{-.063996} \right) \end{aligned}$$

$$= .002328$$

(exact same result!)

$$\begin{aligned} q_{41}^{(w)} &= q_{41}^{(T)} - q_{41}^{(d)} \\ &= .06199 - .002328 \\ &= .059662 \end{aligned}$$

$$\begin{aligned} q_{40}^{(w)} &= q_{40}^{(T)} - q_{41}^{(d)} \\ &= .06326 - .00213 \\ &= .06113 \end{aligned}$$

$$\begin{aligned} q_{40}^{(w)} + p_{40}^{(T)} \cdot q_{41}^{(w)} &= .06113 + .93674(.059662) \\ &= .11702 \end{aligned}$$

(C)

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15. This problem is somewhat similar to 1993 #18 and 1996 #24. In those problems, all new entrants occurred at a single age.

The information given in this problem is partly designed to confuse you. L_x is rarely used in population problems.

T_x can be interpreted as the number of lives at and above age x at any moment in time. It can also be interpreted as the number of years to be lived from age x until death by the L_x people attaining age x in a single year.

For this problem, we'll use the first definition of T_x . T_{18} represents the total population based on L_{18} new entrants. Since there are 4000 new entrants at age 18 each year, the stationary population would be $\left(\frac{T_{18}}{L_{18}}\right)(4,000) = \frac{266,668}{100,000}(4,000) = 10,667$

The shortfall between the 10,667 and the final 15,000 number in the stationary population is $4,333 = 15,000 - 10,667$. The new admissions at age 19 must produce the additional 4,333 members:

$$\left(\frac{T_{19}}{L_{19}}\right)Z = 4,333$$

$$Z = 4,333 \left(\frac{L_{19}}{T_{19}}\right) = 4,333 \left(\frac{93,750}{168,751}\right) = 2,407$$

(B)

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16. The key to this problem is knowing what is implied by "if only Smith is alive". This means that a reversionary annuity is paid to Smith, after Brown's death.

I'll represent a life annuity to Smith as a_s , and a life annuity to Brown as a_B . Now I can write down the expressions for the various annuity values

$$a_s = 100$$

$$a_B = X$$

$$a_s - a_{sB} = 20$$

$$(a_s - a_{sB}) + (a_B - a_{sB}) = 50$$

Now substitute values to solve for a_{sB} , then for a_B :

$$a_s - a_{sB} = 20 = 100 - a_{sB} \quad \therefore a_{sB} = 80$$

$$(a_s - a_{sB}) + (a_B - a_{sB}) = 50$$

$$(100 - 80) + (a_B - 80) = 50 \quad \therefore a_B = 110 = X$$

(D)

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17. This problem is relatively straightforward, once you set up the definition of the premium

$$\begin{aligned} P &= 600 \ddot{a}_{65}^{(12)} + 10,000 A_{65:10}^1 \\ &= 600 \left(\ddot{a}_{65} - \frac{11}{24} \right) + 10,000 \left(\frac{M_{65} - M_{75}}{D_{65}} \right) \end{aligned}$$

With no M_x values, use the relationship

$$M_x = D_x - d \cdot N_x$$

$$d = i v = .07 / 1.07 = .06542$$

$$\begin{aligned} P &= 600 \left(\frac{N_{65}}{D_{65}} - \frac{11}{24} \right) + 10,000 \left(\frac{D_{65} - d \cdot N_{65} - D_{75} + d \cdot N_{75}}{D_{65}} \right) \\ &= 600 \left(\frac{8872}{965} - \frac{11}{24} \right) + 10,000 \left(\frac{965 - .06542(8872 - 2379) - 346}{965} \right) \\ &= 600(8.7354) + 10,000(.2013) \\ &= 7253.93 \end{aligned}$$

(B)

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15. The key point to solving this problem is knowing how a life traverses the select+ultimate table. For someone age $[68]+2$, the path is as follows

Age	Select	0	1	2	Ultimate
68		$[68]$			
69			$[68]+1$		
70				$[68]+2$	
71					71

Based on the information given, the select period is three years. You can calculate the ultimate l_x and p_x and q_x values at any age. You also can calculate q_{x+n} based on the formula given in the problem.

You can derive the value of $l_{[68]+2}$ based on the relationship $\frac{l_{71}}{l_{[68]+2}} = p_{[68]+2} = 1 - (q_{[68]+2})$

$q_{[68]+2} = .90(q_{70})$ based on the table given in problem

$$l_{70} = 300 \quad l_{71} = 290 \quad q_{70} = \frac{10}{300} = .03333\bar{3}$$

$$= 1000 - 10(70) \quad = 1000 - 10(71)$$

$$q_{[68]+2} = .90(.03333\bar{3}) = .030$$

$$l_{[68]+2} = \frac{l_{71}}{p_{[68]+2}} = \frac{290}{1-.03} = 298.97$$

(D)

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19. The definition of curtate expectation of life is similar to an annuity immediate with no interest:

$$a_x = v p_x + v^2 {}_2p_x + \dots \quad e_x = p_x + {}_2p_x + \dots$$

$$e_0 = p_0 + {}_2p_0 + {}_3p_0 + \dots$$

Age x	q_x	p_x
0 to 35	.01/1.01	1/1.01
36 to 75	.02/1.02	1/1.02
> 75	1	0

The definitions of q_x are slightly confusing, since now you must calculate the sum of probabilities of survival from age 0 to age 75. Since $q_{76} = 1$, no one survives to age 77.

$$\begin{aligned}
 e_0 &= \frac{1}{(1.01)^1} + \frac{1}{(1.01)^2} + \dots + \frac{1}{(1.01)^{35}} + \frac{1}{(1.01)^{36}} \left[1 + \frac{1}{(1.02)^1} + \frac{1}{(1.02)^2} + \dots + \frac{1}{(1.02)^{40}} \right] \\
 \text{Survive to Age} \quad & \quad \quad 1 \quad \quad 2 \quad \quad \dots \quad 35 \quad \quad \quad 36 \quad 37 \quad 38 \quad \dots \quad 76 \\
 &= a_{\overline{35}|.01} + (1.01)^{-36} a_{\overline{41}|.02} \\
 &= a_{\overline{35}|.01} + (1.01)^{-36} (1.02) a_{\overline{41}|.02} \quad (\text{all immediate annuities}) \\
 &= 29.4086 + .6989 (1.02) (27.7995) \\
 &= 49.2262
 \end{aligned}$$

The tricky and confusing part is being sure you have the right number of terms, and handle the cross-over at age 36 correctly. (B)

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- 20 There are two alternate solution methods for this problem. This page shows the shortest solution, and the next page has a longer alternative.

This is essentially a test of your knowledge of various identities. You are given values for N_x and D_x , as well as an insurance reserve. The best identity to use is this: ${}_tV_x = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$

$${}_5V_{40} = 1 - \frac{\ddot{a}_{45}}{\ddot{a}_{40}} \Rightarrow \frac{\ddot{a}_{45}}{\ddot{a}_{40}} = 1 - {}_5V_{40} \Rightarrow \ddot{a}_{40} = \frac{\ddot{a}_{45}}{1 - {}_5V_{40}}$$

Now you have expressed \ddot{a}_{40} in terms of information you have been given. You can directly calculate the value of $1000 {}_6V_{40}$ using a similar identity:

$$\begin{aligned} {}_6V_{40} &= 1 - \frac{\ddot{a}_{46}}{\ddot{a}_{40}} = 1 - (N_{46}/D_{46}) \left(\frac{1 - {}_5V_{40}}{\ddot{a}_{45}} \right) \\ &= 1 - (N_{46} - D_{45})/D_{46} \left(\frac{1 - .04346}{N_{45}/D_{45}} \right) \\ &= 1 - (536,152/42,362) \left(\frac{.95654}{581,634/45,482} \right) \\ &= .05332 \end{aligned}$$

$$1000 {}_6V_{40} = 53.32$$

(E)

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(20) (Alternate longer method of solution)

The key to this problem is figuring how to work it with so little information.

You need to use the identity $A_x = 1 - d \ddot{a}_x$, which allows you to simplify things a bit:

$$\begin{aligned} 1000 {}_5V_{40} &= 43.46 = PV(\text{Future benefits}) - PV(\text{Future premiums}) \\ &= 1000(A_{45} - P_{40} \ddot{a}_{45}) \\ &= 1000(1 - d \ddot{a}_{45} - P_{40} \ddot{a}_{45}) \\ &= 1000(1 - \ddot{a}_{45}(P_{40} + d)) \\ .04346 &= 1 - \ddot{a}_{45}(P_{40} + d) \\ P_{40} + d &= \frac{.95654}{\ddot{a}_{45}} = \frac{45,482(.95654)}{581,634} \\ &= .07480 \end{aligned}$$

$$\begin{aligned} 1000 {}_6V_{40} &= 1000(A_{46} - P_{40} \ddot{a}_{46}) \\ &= 1000(1 - \ddot{a}_{46}(P_{40} + d)) \text{ similar to prior formula} \\ &= 1000(1 - \frac{N_{46}}{D_{46}}(.07480)) \\ &= 1000\left(1 - \frac{(581,634 - 45,482)(.07480)}{42,362}\right) \\ &= 1000(1 - 12.6564(.07480)) \\ &= 53.32 \end{aligned}$$

(E)

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- 21 you can calculate the present value of the monthly life annuity based on the given information:

$$\begin{aligned} 12,000 a_{40}^{(12)} &= 12,000 (\ddot{a}_{40} - 13/24) \\ &= 12,000 \left(\frac{8,700}{651} - \frac{13}{24} \right) \\ &= 153,869 \end{aligned}$$

The present value of the perpetuity equals this amount of 153,869, since the two annuities are actuarially equivalent. You need to determine the interest rate in order to solve for the monthly payment P :

$$\begin{aligned} \frac{D_{41}}{D_{40}} &= v p_{40} = \frac{1 - p_{40}}{1+i} \Rightarrow 1+i = \frac{D_{40}(1-p_{40})}{D_{41}} \\ &= \frac{651(1-.002125)}{607} \\ &= 1.07021 \end{aligned}$$

The resulting interest rate is not quite 7%!! Now you can write the formula for the PV of the perpetuity:

$$\begin{aligned} PV = 153,869 &= P(1.07021)^{-\frac{1}{12}} + P(1.07021)^{-\frac{2}{12}} + P(1.07021)^{-\frac{3}{12}} + \dots \\ (1.07021)^{\frac{1}{12}}(153,869) &= P(1.07021)^{-\frac{1}{12}} + P(1.07021)^{-\frac{1}{12}} + P(1.07021)^{-\frac{1}{12}} + \dots \\ (1.07021)^{\frac{1}{12}}(153,869) &= P(1.07021)^{-\frac{1}{12}} \\ P &= 153,869 \frac{(1 - (1.07021)^{-\frac{1}{12}})}{(1.07021)^{-\frac{1}{12}}} = 153,869 \frac{(0.005639)}{.994361} \\ &= 872.51 \end{aligned}$$

If you simply use 1.07000, the value of P is 870.00

(B)

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- 22 This problem has appeared on the exam before. It involves use of this identity which relates successive annual life annuity values:

$$v p_x \ddot{a}_{x+1} = a_x = \ddot{a}_x - 1.0$$

The messy part of the problem is the data you have to work with - monthly annuities instead of annual annuities. If you don't adjust the values properly, you get the wrong answer.

$$\begin{aligned} p_x &= \frac{(\ddot{a}_x - 1.0)(1+i)}{\ddot{a}_{x+1}} & \ddot{a}_x^{(12)} &= \ddot{a}_x - \frac{11}{24} \\ &= \frac{(\ddot{a}_x^{(12)} + \frac{11}{24} - \frac{24}{24})(1+i)}{\ddot{a}_{x+1}^{(12)} + \frac{11}{24}} & &= (1+i) \left(\frac{\ddot{a}_x^{(12)} - 13/24}{\ddot{a}_{x+1}^{(12)} + 11/24} \right) \end{aligned}$$

The problem asks for $1/q_x = p_x(q_{x+1}) = p_x(1 - p_{x+1})$. You can use the modified expression above to directly calculate the p_x values from $\ddot{a}_x^{(12)}$ and $\ddot{a}_{x+1}^{(12)}$.

$$p_x = \frac{1.07(7.6022 - 13/24)}{(7.3683 + 11/24)} = .9653$$

$$p_{x+1} = \frac{1.07(7.3683 - 13/24)}{(7.1321 + 11/24)} = .9623$$

$$\begin{aligned} 1/q_x &= p_x(1 - p_{x+1}) \\ &= .9653(1 - .9623) \\ &= .03636 \end{aligned}$$

(B)

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23. This is a fairly straightforward question. The only trick is that you are given $N_x^{(12)}$ values instead of N_x , and the premiums are paid annually.

$$\ddot{a}_x^{(12)} = \ddot{a}_x - \frac{11}{24}$$

$$\frac{N_x^{(12)}}{D_x} = \frac{N_x}{D_x} - \frac{11}{24}$$

$$N_x^{(12)} = N_x - \frac{11}{24} D_x$$

$$P \cdot \ddot{a}_{40:\overline{25}|} = 12,000 \ddot{a}_{65}^{(12)} \frac{D_{65}}{D_{40}}$$

$$P = \frac{12,000 N_{65}^{(12)} / D_{40}}{(N_{40} - N_{65}) / D_{40}}$$

$$= \frac{12,000 N_{65}^{(12)}}{N_{40} - N_{65}}$$

$$= \frac{12,000 N_{65}^{(12)}}{N_{40}^{(12)} + \frac{11}{24} D_{40} - \left(N_{65}^{(12)} + \frac{11}{24} D_{65} \right)}$$

$$= \frac{12,000 (4,320)}{52,051 + \frac{11}{24} (4,454) - \left[4,320 + \frac{11}{24} (527) \right]}$$

$$= 1046.62$$

(B)

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24. This problem requires knowledge of the formula for the expectation of life

$$e_x = p_x + 2p_x + 3p_x + \dots$$

You also need to use the relationship between values of e_x at successive ages. It is similar to the identity for life annuities:

$$v p_x \ddot{a}_{x+1} = \ddot{a}_x - 1.0$$

$$v p_x (a_{x+1} + 1.0) = a_x$$

$$p_x (e_{x+1} + 1.0) = e_x$$

The problem asks for the probability that someone age 63 will not survive to age 65, which is $1 - {}_2p_{63}$.

$$\begin{aligned} p_{63} &= \frac{e_{63}}{1 + e_{64}} \\ &= 9.5/10 \end{aligned}$$

$$\begin{aligned} p_{64} &= \frac{e_{64}}{1 + e_{65}} \\ &= 9.0/9.5 \end{aligned}$$

$$\begin{aligned} 1 - {}_2p_{63} &= 1 - p_{63}(p_{64}) \\ &= 1 - \frac{9.5}{10} \left(\frac{9.0}{9.5} \right) \\ &= 1 - \frac{9}{10} \\ &= .10 \end{aligned}$$

(B)

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25. You need to write down the "first principles" expression for whole life insurance at age 50 to derive the relationship for the value at age 53:

$$A_{50} = v f_{50} + v^2 p_{50} f_{51} + v^3 {}_2p_{50} f_{52} + v^4 {}_3p_{50} f_{53} + \dots$$

$$A_{53} = v f_{53} + v^2 p_{53} f_{54} + \dots$$

$$A_{50} = v f_{50} + v^2 p_{50} f_{51} + v^3 {}_2p_{50} f_{52} + v^3 {}_3p_{50} [A_{53}]$$

$$A_{53} = \frac{A_{50} - v f_{50} - v^2 p_{50} f_{51} - v^3 {}_2p_{50} f_{52}}{v^3 {}_3p_{50}}$$

You are given that $10,000 A_{50} = 5,000$

$$\begin{aligned} 10,000 A_{53} &= \frac{5,000 - 10,000(v d_{50} + v^2 d_{51} + v^3 d_{52})/l_{50}}{v^3 \frac{l_{53}}{l_{50}}} \\ &= \frac{5000 - 10,000 \left(\frac{5}{1.08} + \frac{5}{(1.08)^2} + \frac{5}{(1.08)^3} \right) / 100}{(1.08)^{-3} (85) / (100)} \\ &= \frac{5000 - 10,000 (4.63 + 4.29 + 3.97) / 100}{.6748} \\ &= \frac{5000 - 1289}{.6748} \\ &= 5580 \end{aligned}$$

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