SoftwarePolish

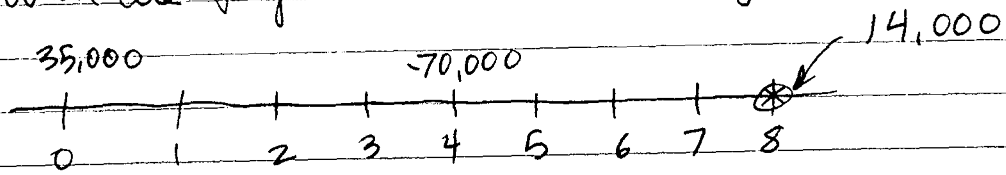
Rick Groszkiewicz
2964 Nestle Creek Drive
Marietta, GA 30062-4857

Voice/fax (770) 971-8913
email: rickg@softwarepolish.com

SPRING 1998 EA-1A EXAM SOLUTIONS

Spring 1998 EA-1A

1 The key to this problem is correctly writing down the payments on a time diagram:



$$35{,}000(1+i)^8 - 70{,}000(1+i)^4 = 14{,}000$$

Once you have this relationship, there are two ways to get the answer. One is to simply test the answer ranges. If you start with 14%, then calculate the left side of the above equation, you'll get a negative result. Next try 19%, which produces a small positive result. Finally, 24% will give too large of a positive result, so the answer is in range C, between 19% and 24%.

The alternative is to rewrite the equation by substituting $z = (1+i)^4 \Rightarrow 35{,}000z^2 - 70{,}000z - 14{,}000 = 0$

divide by 7,000 $\quad 5z^2 - 10z - 2 = 0$

Now apply the quadratic formula, which gives the answer to $az^2 + bz + c = 0$ as $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$$z = \left(10 \pm \sqrt{(10)^2 - 4(5)(-2)}\right)/2(5)$$
$$= \left(10 \pm \sqrt{100 + 40}\right)/10$$
$$= \left(10 + \sqrt{140}\right)/10 \quad \text{or a negative result - ignore that one!}$$
$$= 2.1832$$
$$\therefore (1+i)^4 = 2.1832 \Rightarrow 1+i = 1.2156$$

C

This is one of MANY problems on this exam solved by the quadratic formula.

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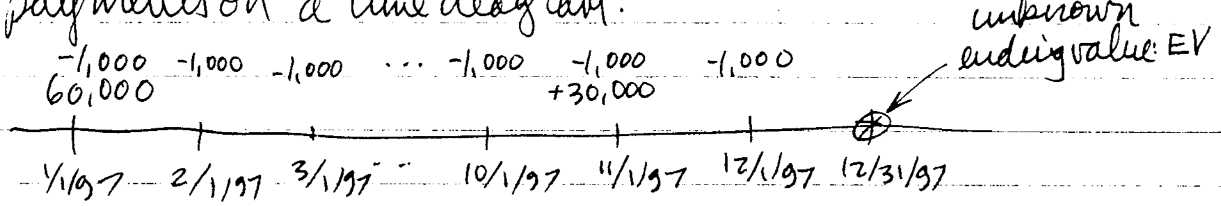
2. The key to this problem is your ability to write down the formula for the dollar weighted interest rate calculation. First, you can show the payments on a time diagram:



$$60{,}000(1.07) + 30{,}000\left(1+\frac{.07}{12}(2)\right) - 1{,}000\left(12+\frac{.07}{12}(12+11+\ldots+1)\right) = EV$$

this can be re-written:

$$60{,}000(1+.07) + 30{,}000\left(1+\frac{.07}{12}(2)\right) - 12{,}000\left(1+.07\left(\frac{13}{24}\right)\right)^{*} = EV$$

Now, based on the 6% expected return, you would have

$$60{,}000(1+.06) + 30{,}000\left(1+\frac{.06}{12}(2)\right) - 12{,}000\left(1+.07\left(\frac{13}{24}\right)\right) = \text{expected value}$$

The difference between the two equations gives the investment gain for 1997:

$$60{,}000(.01) + 30{,}000\left(\frac{.01}{12}(2)\right) - 12{,}000\left(.01\left(\frac{13}{24}\right)\right)$$

$= 600 + 50 - 65$

$= 585$ (A) within "implied range" 540-590

* Derivation of $1+(13/24)i$ factor:

$1{,}000\left(12 + (.07/12)(12+11+\ldots+1)\right)$

$= 1{,}000\left(12 + (.07/12)\left(\frac{12(13)}{2}\right)\right)$

$= 1{,}000\left(12 + .07(13/2)\right)$

$= 12{,}000\left(1 + .07(13/24)\right)$

I usually remember this 13/24 factor for monthly benefit payments at the beginning of each month, simple interest.

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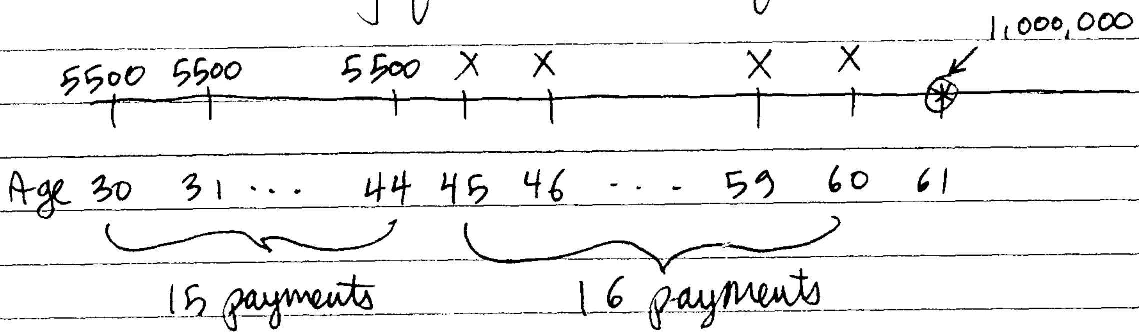
3 The key to this problem is carefully writing down the payments on a time diagram, and then solving for the value of X:



One point of the problem is correctly reading that there are 31 total payments!

$$5500\left(\ddot{s}_{\overline{31}|.09} - \ddot{s}_{\overline{16}|.09}\right) + X\,\ddot{s}_{\overline{16}|.09} = 1{,}000{,}000$$

$$X = \frac{1{,}000{,}000 - 5500\left(\ddot{s}_{\overline{31}|} - \ddot{s}_{\overline{16}|.09}\right)}{\ddot{s}_{\overline{16}|.09}}$$

$$= \frac{1{,}000{,}000 - 5500(1.09)\left(s_{\overline{31}|.09} - s_{\overline{16}|.09}\right)}{1.09\, s_{\overline{16}|.09}}$$

$$= \frac{1{,}000{,}000 - 5500(1.09)(149.5752 - 33.0034)}{1.09(33.0034)}$$

$$= \frac{301{,}152}{35.9737}$$

$$= 8{,}371.44 \qquad \text{C}$$

The reason for writing the solution using immediate annuities instead of due is that I try to avoid changing that setting on the calculator. By always using annuities immediate, I avoid some arithmetic errors caused by carelessness.

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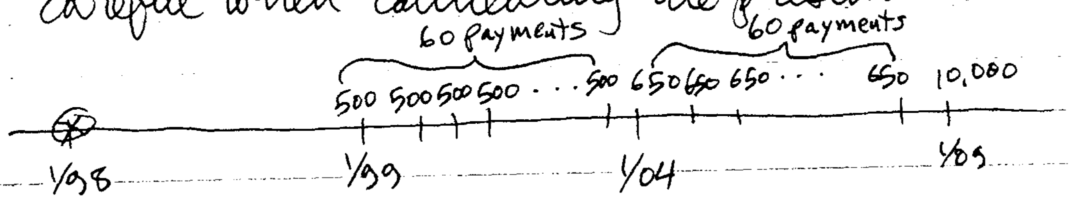
4. The key to this problem is writing down the payments on a time diagram, and being careful when calculating the present value



I'll use a monthly interest rate j to calculate the value of the monthly payments:

$$(1+j)^{12} = 1.07$$
$$j = .565\%$$

$$1\text{-}1\text{-}98\ PV = (1.07)^{-\frac{11}{12}}\left(500\, a_{\overline{60}|.565} + (1.07)^{-5}\, 650\, a_{\overline{60}|.565}\right) + 10{,}000(1.07)^{-11}$$
$$= (1.00565)^{-11}\left(500\left(a_{\overline{60}|.565}\right) + (1.07)^{-5}650\left(a_{\overline{60}|.565}\right)\right) + 10{,}000(1.07)^{-11}$$
$$= (1.00565)^{-11}\left(a_{\overline{60}|.565}\right)\left(500 + (1.07)^{-5}(650)\right) + 10{,}000(1.07)^{-11}$$
$$= .9399\,(50.7617)(963.4410) + 10{,}000\,(.4751)$$
$$= 50{,}715.78$$

(B)

An alternative approach would use $a^{(12)}_{\overline{5}|.07}$ with an annual payment of 6000. The results are identical. I prefer to convert the interest rate to match the frequency of the payment instead.

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5 The key to this problem is correctly determining the outstanding loan balance at 1/1/07. The loan interest in the 12/31/07 repayment will be 7% of that outstanding loan balance.

The first step is writing down the loan repayments on a time diagram:

1/98	12/98	12/99	12/00	12/01	12/02	12/03	12/04	12/05	12/06	12/07	12/08	...	12/11	12/12
	1000	1000	1000	1000	1000	-0-	-0-	3000	1000	1000	1000	...	1000	OSB

The way the loan repayments are stated, the loan repayment "for a year" corresponds to the loan repayment on the last day of the year. That is why you need to determine the outstanding loan balance at 1/1/07.

There are normally two equivalent methods to find the outstanding loan balance. You can't use the prospective method, since the final payment is unknown. Under the retrospective method, subtract the accumulated past payments from the accumulated original loan amount:

$$\begin{aligned}
1/1/07 \text{ loan balance} &= 10{,}000(1.07)^9 - \left[1000\left(s_{\overline{9}|.07} - s_{\overline{4}|.07}\right) + 3000(1.07) + 1000\right] \\
&= 18{,}384.59 - \left[1000(11.9780 - 4.4399) + 3{,}210 + 1{,}000\right] \\
&= 18{,}384.59 - 11{,}748 \\
&= 6{,}636.55
\end{aligned}$$

Interest paid for year 2007 $= .07(6636.55) = 464.56$ C

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6 The key to this problem is using Makeham's formula, which simplifies serial bond problems quite a bit. You will get the same answer with other bond price formulas, but it will take more effort.

With a serial bond, you have multiple redemption dates. In general, the bond is "spread" between the redemption dates equally.

Makeham's formula $\quad P = K + (g/i)(C-K)$

$$= K + \frac{Fr}{Ci}(C-K)$$

with a single redemption date: $P = K + \frac{10{,}000(.07)}{10{,}000(.12)}(10{,}000-K)$

$$= K + (7/12)(10{,}000-K)$$

With a serial bond, K represents the sum of K, which is Cv^n, for all the redemption dates:

$$K = (10{,}000)\tfrac{1}{20}(v^1 + v^2 + \ldots + v^{20})$$
$$= 500\, a_{\overline{20}|.12}$$
$$= 3734.72$$
$$P = 3734.72 + (7/12)(10{,}000 - 3734.72)$$
$$= 3734.72 + .58333(6265.28)$$
$$= 7{,}389.47$$

(A) within "implied range" 7000-7500

You will arrive at the same result if you use Makeham's formula for each of the 20 redemption dates separately - eventually you'll be comfortable calculating K directly as shown above.

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7 This is an identity problem that is either easy or hard, depending on whether you see how to get to the answer. The key is getting both annuities with the same period. Then the ratio will give you the "v^n" term that is used in either annuity value.

$$a_{\overline{n+1}|} = \ddot{a}_{\overline{n+2}|} - 1.0 = 13.030$$
$$s_{\overline{n+1}|} = \ddot{s}_{\overline{n}|} + 1.0 = 53.344$$

$$\frac{a_{\overline{n+1}|}}{s_{\overline{n+1}|}} = v^{n+1} = \frac{13.030}{53.344} = .2443$$

$$a_{\overline{n+1}|} = \frac{1-v^{n+1}}{i} = \frac{1-.2443}{i} = 13.030$$

$$i = \frac{1-.2443}{13.030} = 5.80\%$$ E

within "implied range" 5.75% to 6.00%

Alternate solution based on iteration:

$s_{\overline{n}|} = v(52.344)$ $\quad a_{\overline{n+1}|} = 13.030$ $\quad \ddot{a}_{\overline{n+1}|} = (1+i)\,13.030$

$$a_{\overline{n}|} = (1+i)(13.030) - 1$$

$$\frac{a_{\overline{n}|}}{s_{\overline{n}|}} = v^n = \frac{(1+i)(13.030)-1}{v(52.344)} \qquad \frac{1-v^n}{i} = (1+i)(13.030)-1$$

$$v^n = 1 - i\left[(1+i)(13.030)-1\right]$$

$$1 - i\left[(1+i)13.030 - 1\right] = \frac{\left[(1+i)(13.030)-1\right]}{v(52.344)}$$

$$i\left[(1+i)13.030 - 1\right] = 1 - \frac{\left[(1+i)(13.030)-1\right]}{v(52.344)}$$

$$i = \frac{1}{(1+i)(13.030)-1} - \frac{1+i}{52.344}$$

1st guess $i_1 = .05$ produces $i_2 = .0588 \Rightarrow i_3 = .0579 \Rightarrow i_4 = .0580$

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8 The key to this problem is setting up the loan amortization schedule for the first 24 payments.

Assume the monthly payment is P. Then set up the amortization schedule:

Date	Payment number	Payment amount	Interest paid	Principal paid	Outstanding loan amount
	0				$P\,a_{\overline{36}\vert 1.5\%}$
1/31/96	1	P	$P(1-v^{36})$	Pv^{36}	$P\,a_{\overline{35}\vert 1.5\%}$
2/28/96	2	P	$P(1-v^{35})$	Pv^{35}	⋮
	⋮	⋮	⋮	⋮	
12/31/97	24	P	$P(1-v^{13})$	Pv^{13}	$P\,a_{\overline{12}\vert 1.5\%}$

Interest paid for 1st 24 payments: $P(1-v^{36}+1-v^{35}+\ldots+1-v^{13})$

$= P(24 - v^{36} - v^{35} - \ldots - v^{13})$

$= P(24 - a_{\overline{36}\vert} + a_{\overline{12}\vert})$

Principal paid for 1st 24 payments $= 24P - \text{Interest paid}$

$= P(a_{\overline{36}\vert} - a_{\overline{12}\vert})$

$$\text{Ratio} = \frac{24-(a_{\overline{36}\vert}-a_{\overline{12}\vert})}{a_{\overline{36}\vert}-a_{\overline{12}\vert .015}}$$

$$= \frac{24-(27.6607-10.9075)}{27.6607-10.9075}$$

$$= \frac{24-16.7532}{16.7532}$$

$= .4326$ C

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9. The key to this problem is reading it carefully. It sounds much worse than it actually is!

For the original loan, the outstanding balance at time t is $10{,}000\, a_{\overline{30-t}|.07}$, which is the present value of the outstanding loan payments.

At time 7, the O/S balance is $10{,}000\, a_{\overline{23}|.07}$
$= 112{,}722$

Now the loan is increased by 50,000 to 162,722. With 14 annual repayments, the annual repayment can be calculated.

$$P\, a_{\overline{14}|.07} = 162{,}722$$
$$P = 18{,}606$$

One year later, the outstanding loan balance is $P\, a_{\overline{13}|.07}$. The interest paid in the second installment is $.07\,[P\, a_{\overline{13}|.07}]$

$$= .07\,[18606]\left(\frac{1-v^{13}}{.07}\right)$$
$$= 18{,}606\,(1-v^{13})$$
$$= 10{,}885$$

C

You could also set up a detailed loan amortization schedule if you wanted to be <u>very</u> careful. See the prior problem for an example of this.

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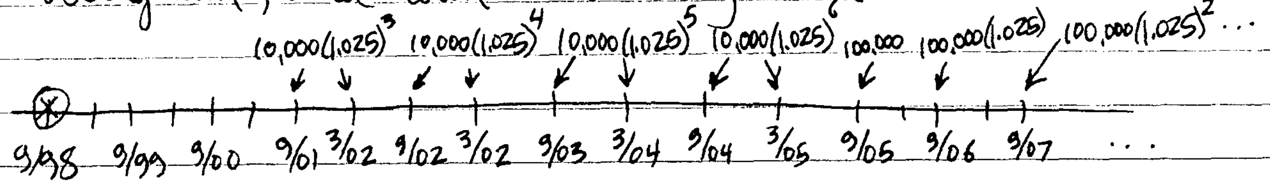
10 The key to this problem is carefully writing down the sequence of payments on a time diagram, and then evaluating them:



Tuition 1

$$PV = (1.08)^{-3}\,10{,}000\left[(1.025)^3 + (1.08)^{-1}(1.025)^4 + (1.08)^{-2}(1.025)^5 + (1.08)^{-3}(1.025)^6\right]\left(1+(1.08)^{-.5}\right)$$

The last term allows for the two sets of 4 identical payments that are made at 9/1 and 3/1. Factor out 1st term:

$$PV = (1.08)^{-3}\left(1+(1.08)^{-.5}\right)10{,}000(1.025)^3\left[1 + \frac{1.025}{1.08} + \left(\frac{1.025}{1.08}\right)^2 + \left(\frac{1.025}{1.08}\right)^3\right]$$

$= .7938\,(1.9623)\,10{,}000(1.07689)\left(1 + a_{\overline{3}|j}\right)$ where $1+j = \frac{1.08}{1.025} = 1.05366$

$= 62{,}145$

Tuition 2

$$PV = (1.08)^{-7}\,100{,}000\left(1 + \frac{1.025}{1.08} + \left(\frac{1.025}{1.08}\right)^2 + \dots\right)$$

$= (1.08)^{-7}\,\frac{100{,}000}{d}$ for perpetuity due based on rate j: $1+j = 1.05366$

$= .5835\,\frac{(100{,}000)}{.05366/1.05366}$

$= 1{,}145{,}763$

Total is 1,207,908 (D)

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11 The modified duration is regular duration over $(1+i)$. Regular duration is a weighted average of the year of payment, where the weight is the present value of the payment:

$$\text{modified duration} = \frac{\bar{d}}{1+i} \quad \text{where } \bar{d} = \frac{\sum_{t=1}^{n} t\, v^t R_t}{\sum_{t=1}^{n} v^t R_t}$$

You are given the modified duration for bond 1:

$$6.42 = \frac{1}{1.09}\left(\frac{1 \cdot 90v + 2 \cdot 90v^2 + \dots + 9 \cdot 90v^9 + 10 \cdot 90v^{10} + 10 \cdot 1000v^{10}}{90v + 90v^2 + \dots + 90v^9 + 90v^{10} + 1000v^{10}}\right)$$

$$= \frac{1}{1.09}\left(\frac{90(Ia)_{\overline{10}|.09} + 10{,}000v^{10}}{90a_{\overline{10}|.09} + 1{,}000v^{10}}\right)$$

$$= \frac{1}{1.09}\left(\frac{90(Ia)_{\overline{10}|.09} + 10{,}000v^{10}}{1000}\right)$$

Since the coupon rate and yield rate are both 9% per annum, the purchase price of the bond is the same as the redemption value!

You can solve for $(90(Ia)_{\overline{10}|.09} + 10{,}000v^{10}) = 6.42(1.09)(1000) = 6997.80$. You can use that value to determine the modified duration of the portfolio:

$$\text{Duration} = \frac{1}{1.09}\left(\frac{90(Ia)_{\overline{10}|.09} + 10{,}000v^{10} + 13{,}000v^{13}}{90a_{\overline{10}|.09} + 1{,}000v^{10} + 1{,}000v^{13}}\right)$$

$$= \frac{1}{1.09}\left(\frac{6997.80 + 13{,}000v^{13}}{1000 + 1{,}000v^{13}}\right)$$

$$= \frac{11{,}238.12}{1445.53} = 7.77$$

A written "implied range" of 7.4 to 7.9

(You get the same results if you simply ignore the 6.42 and evaluate above expression)

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12 The key to this problem is interpreting the information about early redemption of the bond. Since 40% of the bond will be redeemed after 5 years, 60% of the bond will be redeemed at the end of 10 years:

$$620 = 320v^5 + .6(1000)v^{10}$$
$$600v^{10} + 320v^5 - 620 = 0$$
$$30v^{10} + 16v^5 - 31 = 0$$

Similarly to question 1, you can either test the answer ranges, or use the quadratic formula. For the latter approach, let $x = v^5$:

$$30x^2 + 16x - 31 = 0$$
$$x = \frac{-16 \pm \sqrt{(16)^2 - 4(30)(-31)}}{2(30)}$$
$$= \frac{-16 \pm \sqrt{256 + 3720}}{60}$$
$$= \frac{-16 \pm \sqrt{3976}}{60}$$
$$= \frac{-16 + 63.0555}{60} \quad \text{(ignore negative result)}$$
$$x = .7843 = v^5$$
$$\therefore v = (.7843)^{.2}$$
$$= .9526$$
$$i = 4.98\% \qquad \text{B}$$

If you read the problem to incorrectly state that there is a 40% probability that the bond is called for 320 after 5 years, you have $620 = .4(320v^5) + 600v^{10}$. This produces an answer of 1.78% which is outside the "implied range"!

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13 There are two ways to work this problem, using various formulas relating the single decrement rates to the multiple decrement probabilities.

One formula that is true for <u>any</u> number of decrements relates the probability of survival:

$$p'^{(d)}_x \doteq \left[p^{(T)}_x\right]^{q^{(d)}_x / q^{(T)}_x}$$

Since $q^{(T)}_x = q^{(d)}_x + q^{(w)}_x$, and you are told that $q^{(w)}_x = 5(q^{(d)}_x)$

then $q^{(d)}_x / q^{(T)}_x = 1/6$

$$p'^{(d)}_x = 1 - .035 = .965 = \left[p^{(T)}_x\right]^{1/6} \Rightarrow p^{(T)}_x = (.965)^6 = .8075$$

Based on similar reasoning for withdrawals, you have

$$p'^{(w)}_x = \left[p^{(T)}_x\right]^{5/6} = (.8075)^{5/6} = .8368$$

$$\therefore q'^{(w)}_x = 1 - .8368 = .1632 \qquad \text{(B)}$$

Another approximation for a double decrement table is

$$q'^{(d)}_x \doteq \frac{q^{(d)}_x}{1 - \frac{1}{2} q^{(w)}_x}$$

$$.035 = \frac{q^{(d)}_x}{1 - \frac{1}{2}(5 q^{(d)}_x)} \Rightarrow q^{(d)}_x = .035\left[1 - 2.5 q^{(d)}_x\right]$$

$$= \frac{.035}{1.0875} = .03218$$

$$q^{(w)}_x = 5 q^{(d)}_x$$
$$= 5(.03218) = .1609$$

$$q'^{(w)}_x \doteq \frac{q^{(w)}_x}{1 - \frac{1}{2} q^{(d)}_x} = \frac{.1609}{1 - .5(.0322)}$$
$$= .1636$$

The apparent difference is due to the data given. The answers only have 2 significant digits of accuracy!

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14 There are two keys to working this problem:
(i) understanding of general probability concepts
(ii) knowledge of the quadratic formula

Assume that Smith and Brown are both age x. The probability that at least one person dies in the next year is equivalent to 1.0 minus the probability none dies: $1-p_x(p_x)$. This is equal to 20 time the probability that both die in the next year:

$$1-p_xp_x = 20(1-p_x)(1-p_x)$$

For clarity, replace p_x with z:

$$1-z^2 = 20(1-2z+z^2)$$
$$= 20-40z+20z^2$$
$$21z^2-40z+19=0$$

Now use the quadratic formula to solve for z

$$z = \frac{40 \pm \sqrt{(40)^2-4(21)(19)}}{2(21)}$$
$$= \frac{40 \pm \sqrt{1600-1596}}{42}$$
$$p_x = z = \frac{40 \pm 2}{42} = \frac{42}{42} \text{ or } \frac{38}{42}$$

Ignore the first result since you are told q_x is NOT zero.
The probability that exactly one life will die the next year is 1.0 minus the probability none dies minus the probability both die:

$$1-p_xp_x-q_xq_x = 1-\left(\frac{38}{42}\right)^2-\left(\frac{4}{42}\right)^2$$
$$= 1-.8186-.0091$$

Another quadratic formula problem! $= .1723$ D

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15 The key to this problem is writing all of the probabilities in terms of l_x values. Then you have to realize how to arrive at the desired answer.

(1) $_{10}p_{25:35:45} = \frac{l_{35}}{l_{25}}\left(\frac{l_{45}}{l_{35}}\right)\left(\frac{l_{55}}{l_{45}}\right) = \frac{l_{55}}{l_{25}} = .770$

(2) $_{5}p_{45:50}\left(_{5}q_{40}\right) = \frac{l_{50}}{l_{45}}\left(\frac{l_{55}}{l_{50}}\right)\left(1-\frac{l_{45}}{l_{40}}\right) = \frac{l_{55}}{l_{45}} - \frac{l_{55}}{l_{40}} = .029$

(3) $_{15}p_{25} = \frac{l_{40}}{l_{25}} = .975$

You need a value for $_{20}p_{25} = \frac{l_{45}}{l_{25}}$

If you divide equation (2) by l_{55}/l_{25}, then you can use equation (3) to get close to the answer:

$$\left(\frac{l_{55}}{l_{45}} - \frac{l_{55}}{l_{40}}\right)\frac{l_{25}}{l_{55}} = \frac{.029}{.770} = .03766 = \frac{l_{25}}{l_{45}} - \frac{l_{25}}{l_{40}}$$

This result contains the answer (inverted). You can use equation (3) to eliminate the second term:

$$\frac{l_{25}}{l_{45}} - \frac{l_{25}}{l_{40}} + \frac{l_{25}}{l_{40}} = .03766 + \frac{1}{.975} = 1.0633 = \frac{l_{25}}{l_{45}}$$

Finally, $\frac{l_{45}}{l_{25}} = \frac{1}{1.0633} = .9405$ D

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16 The key to this problem is recognizing an identity for ${}_n p_x$ and ${}_{n|}q_x$:

$${}_n p_x = {}_{n|}q_x + {}_{n+1|}q_x + {}_{n+2|}q_x + \cdots$$

A verbal interpretation is that the probability of surviving n years equals the probability that you die after n years, which is the sum of the n year deferred probability of death, plus all the subsequent deferred probabilities of death.

The problem requires you to calculate a_{105}:

$$a_{105} = v\,p_{105} + v^2\,{}_2p_{105} + v^3\,{}_3p_{105} + v^4\,{}_4p_{105} + v^5\,{}_5p_{105} + \cdots$$

Since ${}_{5|}q_{105}$ is zero, 110 is the last age of the mortality table, and ${}_5p_{105}$ is zero. Using the identity above:

$${}_4p_{105} = {}_{4|}q_{105} + {}_{5|}q_{105} + \cdots = .0067$$
$${}_3p_{105} = {}_{3|}q_{105} + {}_{4|}q_{105} + \cdots = .0334 = .0067 + .0267$$
$${}_2p_{105} = {}_{2|}q_{105} + {}_{3|}q_{105} + \cdots = .1334 = .0334 + .1000$$
$${}_1p_{105} = {}_{1|}q_{105} + {}_{2|}q_{105} + \cdots = .4001 = .1334 + .2667$$

$$a_{105} = \frac{{}_1p_{105}}{1.08} + \frac{{}_2p_{105}}{(1.08)^2} + \frac{{}_3p_{105}}{(1.08)^3} + \frac{{}_4p_{105}}{(1.08)^4} + \varnothing$$
$$= \frac{.4001}{1.08} + \frac{.1334}{1.1664} + \frac{.0334}{1.2597} + \frac{.0067}{1.3605}$$
$$= .3705 + .1144 + .0265 + .0049$$
$$= .5163$$

$1000\,a_{105} = 516.27$ (A) within "implied range": 350 to 550

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17 The key to this problem is knowledge of an identity that exists between p_x and e_x:

$p_x(1 + e_{x+1}) = e_x$ for a curtate expectation

$\mathring{e}_x \doteq e_x + \frac{1}{2}$

$$p_x = \frac{e_x}{1 + e_{x+1}} \doteq \frac{\mathring{e}_x - \frac{1}{2}}{\mathring{e}_{x+1} + \frac{1}{2}}$$

You can calculate ${}_3q_{50} = 1 - {}_3p_{50} = 1 - p_{50}(p_{51})(p_{52})$ by deriving values for each probability:

$$p_{50} \doteq \frac{\mathring{e}_{50} - \frac{1}{2}}{\mathring{e}_{51} + \frac{1}{2}} = \frac{22.7}{22.9} = .9913 \qquad p_{51} \doteq \frac{\mathring{e}_{51} - \frac{1}{2}}{\mathring{e}_{52} + \frac{1}{2}} = \frac{21.9}{22.2} = .9865 \qquad p_{52} \doteq \frac{\mathring{e}_{52} - \frac{1}{2}}{\mathring{e}_{53} + \frac{1}{2}} = \frac{21.2}{21.4} = .9907$$

$${}_3q_{50} = 1 - .9913(.9865).9907 = .0313$$

E within "implied range" of .031 to .033

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18 The key to working this problem is understanding reversionary annuities, and also correctly interpreting "Smith's portion of the annuity".

The annuity always pays 1200 per year, but the amount is not evenly divided among Smith, Brown and Green. The present value of <u>all</u> payments is $1200\, a_{\overline{xyz}} = 1200\,(a_x + a_y + a_z - a_{xy} - a_{yz} - a_{xz} + a_{xyz})$.

To determine the present value of Smith's payments, you need to consider the cases where Smith is alive:

(1) Smith only alive

Smith receives 1200 per year. The present value formula reflects that both Brown and Green must be dead:

$1200\,(a_x - a_{x:\overline{yz}})$

This annuity pays to x, but only after the last survivor of y and z has died:

$1200\,(a_x - [a_{xy} + a_{xz} - a_{xyz}])$

(2) All three alive

Smith will receive 200 per year

$200\, a_{xyz}$

(3) Smith and Brown only are alive

Smith receives 600 per year $= 200 + \frac{1}{2}(800)$.

This reversionary annuity reflects Green's death:

$600\,(a_{xy} - a_{xyz})$

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(18)

(4) Smith and Green only are alive
Smith receives 300 per year $= 200 + \frac{1}{2}(200)$
The reversionary annuity reflects Brown's death:
$300\,(a_{xz} - a_{xyz})$

Now add all the pieces to get the present value of "Smith's portion of the annuity"

$$
\begin{array}{llllll}
1200\,(& a_x & - a_{xy} & - a_{xz} & + a_{xyz}) \\
200\,(& & & & a_{xyz}) \\
600\,(& & a_{xy} & & - a_{xyz}) \\
300\,(& & & a_{xz} & - a_{xyz}
\end{array}
$$

$\Sigma = 1200\,a_x - 600\,a_{xy} - 900\,a_{xz} + 500\,a_{xyz}$
$= 1200(12) - 600(9) - 900(7) + 500(5)$
$= 14{,}400 - 5{,}400 - 6{,}300 + 2{,}500$
$= 5{,}200$ D

If you write down the corresponding cases for Brown's portion as well as Green's portion, you will get the result previously stated for the PV of all payments: $1200\,a_{\overline{xyz}}$.
I used that as a logical check that all payments have been correctly accounted for!

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19 There are two key points to working this problem: (i) knowledge of basic probability concepts, and (ii) understanding of gross premiums.

Policy A pays 100,000 if at least one person dies during the year. The probability equals 1.0 minus the probability that none die during the year. The net premium is

$$100{,}000\,v\,(1-p_S\,p_B\,p_G)$$

based on p_S = probability that Smith survives, etc.

The gross premium for policy A equals the net premium times 1.07, which allows for the expense loading:

$$8000 = 1.07\left[100{,}000\,v\,(1-p_S\,p_B\,p_G)\right]$$
$$= 100{,}000\,(1-p_S\,p_B\,p_G)$$
$$.08 = 1-p_S\,p_B\,p_G$$
$$.92 = p_S(p_B)p_G$$

Policy B pays 100,000 if at least one of Smith and Green dies during the year. The net premium is

$$100{,}000\,v\,(1-p_S\,p_G)$$

$\therefore$ Gross premium with 7% expense loading is

$$5000 = 1.07(100{,}000\,v\,(1-p_S\,p_G))$$
$$= 100{,}000(1-p_S\,p_G)$$
$$.05 = 1-p_S\,p_G$$
$$.95 = p_S(p_G)$$

The probability Brown survives one year is

$$p_B = \frac{.92}{.95} = .9684$$ C

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20 The key to working this problem is knowledge of reversionary annuities and joint and last survivor annuities. Once you write down the expression for the present value of each annuity, you have an algebra problem to solve.

Represent the annuity for Smith as a_S, Brown's annuity as a_B, and the joint life annuity as a_{SB}.

Option A: $34{,}000 = 2000\, a_{\overline{SB}} = 2000(a_S + a_B - a_{SB})$

Option B: $50{,}000 = 4000\, a_{SB} + 2000(a_B - a_{SB}) + 2000(a_S - a_{SB})$

Option C: $C = 5000\, a_{SB} + 3000(a_B - a_{SB}) + 3000(a_S - a_{SB})$

The expressions for the present values of options B and C use the reversionary annuity $(a_S - a_{SB})$ to represent payments while only Smith is alive (after Brown's death). While both lives survive, there are no net payments from this combination of annuities.

Now you can simplify both options B and C:

B: $50{,}000 = 2000\, a_B + 2000\, a_S$

C: $C = 3000\, a_B + 3000\, a_S - 1000\, a_{SB}$

Combine B and A to solve for the value of a_{SB}:

$34{,}000 = 2000(a_S + a_B - a_{SB}) = 50{,}000 - 2{,}000\, a_{SB} \Rightarrow a_{SB} = 8.0$

$C = 3000\, a_B + 3000\, a_S - 1000\, a_{SB}$

$= \frac{3}{2}(50{,}000) - 1000(8.0)$

$= 75{,}000 - 8{,}000 = 67{,}000$ B

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21 There are two key parts of this problem
(i) calculation of 12/31/97 contributions plus interest, and
(ii) calculation of the contribution refund death benefit.

The first employee contribution was 5% of 1988 pay at 12/31/88. There are nine more contributions:

$.05(35{,}000)$	$.05(35{,}000)(1.04)^1$		$.05(35{,}000)(1.04)^8$	$.05(35{,}000)(1.04)^9$
12/88	12/89	. . .	12/96	12/97

$$12/31/97 \text{ EECWI} = .05(35{,}000)(1.04)^9 + .05(35{,}000)(1.04)^8(1.07)^1 + \dots + .05(35{,}000)(1.04)^1(1.07)^8 + .05(35{,}000)(1.07)^9$$

Now factor out the first term to evaluate the series

$$12/31/97 \text{ EECWI} = .05(35{,}000)(1.04)^9\left[1 + \frac{1.07}{1.04} + \dots + \left(\frac{1.07}{1.04}\right)^8 + \left(\frac{1.07}{1.04}\right)^9\right]$$

$$= 1750(1.04)^9 s_{\overline{10}|j} \quad \text{where } 1+j = \frac{1.07}{1.04} = 1.0288$$

$$= 1750(1.4233)(11.4031)$$

$$= 28{,}403$$

Future death benefits will be based on this value increasing with 7% per annum to the date of death. There are no future employee contributions.

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$$\text{PV death benefit} = 28{,}403(1.07)^{.5}v^{.5}(d_{45}/l_{45}) + 28{,}403(1.07)^{1.5}v^{1.5}(d_{46}/l_{45}) + \dots + 28{,}403(1.07)^{19.5}v^{19.5}(d_{64}/l_{45})$$

$$= 28{,}403(d_{45} + d_{46} + \dots + d_{64})/l_{45}$$

$$= 28{,}403(l_{45} - l_{65})/l_{45}$$

$$= 28{,}403(1 - l_{65}/l_{45})$$

$$= 28{,}403(1 - v^{65}l_{65}(1+i)^{20}/(v^{45}l_{45}))$$

$$= 28{,}403(1 - (1.07)^{20}D_{65}/D_{45})$$

$$= 28{,}403(1 - 5.4274(944/4450)) = 5{,}087 \quad \text{B}$$

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22 The key to working this problem is knowing the formula for a_x given that l_x follows De Moivre's law. Since the l_x formula is a straight line ending at age 100, the mortality table satisfies De Moivre's law, and $\omega = 100$.

The formula for a_x is $\dfrac{n - \ddot{a}_{\overline{n}|}i}{ni}$ where $n = \omega - x$.

$$a_{70} = \frac{30 - \ddot{a}_{\overline{30}|.06}}{30(.06)}$$

$$= \frac{30 - 1.06(13.7648)}{30(.06)}$$

$$= 8.5607 \quad \text{(B)}$$

An alternate solution uses another formula that is true under De Moivre's law: $A_x = \dfrac{a_{\overline{n}|}}{n}$ where $n = \omega - x$:

$$A_{70} = \frac{a_{\overline{30}|.06}}{30} = .4588$$

$$A_{70} = 1 - d\ddot{a}_{70} = 1 - iv(1 + a_{70})$$

$$iv\ddot{a}_{70} = 1 - A_{70}$$

$$\ddot{a}_{70} = \frac{1 - A_{70}}{iv} = \frac{.5412}{.06/1.06} = 9.5607$$

$$\therefore \; a_{70} = 8.5607$$

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23 This is an extremely tedious, messy problem that is probably the hardest one on this exam. It is similar to 1992 #3 and 1995 #21 in concept.

The annual contribution to the saving account is $\frac{100{,}000}{\ddot{s}_{\overline{5}|.07}} = 1625.147$. At the end of year t, the saving account would have a balance of $1625.147(\ddot{s}_{\overline{t}|.07})$. The decreasing death benefits for the term insurance policy would look like this

0	1	2	3	4	5
	$100{,}000 - 16251.47\ddot{s}_{\overline{1}|}$	$100{,}000 - 16251.47\ddot{s}_{\overline{2}|}$	$100{,}000 - 16{,}251.47\ddot{s}_{\overline{3}|}$	$100{,}000 - 16251.47\ddot{s}_{\overline{4}|}$	$100{,}000 - 16251.47\ddot{s}_{\overline{5}|}$

The death benefit for the last year is actually zero, but having this term makes it possible to calculate the premium based on only the N_x values you are given:

$$P\,\ddot{a}_{40:\overline{5}|} = 100{,}000\,A^{1}_{40:\overline{5}|} - 16{,}251.47\left[\frac{v\,d_{40}\ddot{s}_{\overline{1}|}}{l_{40}} + \frac{v^2 d_{41}\ddot{s}_{\overline{2}|}}{l_{40}} + \dots + \frac{v^5 d_{44}\ddot{s}_{\overline{5}|}}{l_{40}}\right]$$

Since $v^t\ddot{s}_{\overline{t}|} = \ddot{a}_{\overline{t}|}$, and $\ddot{a}_{\overline{t}|} = (1-v^t)/d$, you have this form for the summation:

$$\frac{16{,}251.47}{d}\left[\frac{d_{40}}{l_{40}} + \frac{d_{41}}{l_{40}} + \frac{d_{42}}{l_{40}} + \dots + \frac{d_{44}}{l_{40}} - \frac{(v\,d_{40} + v^2 d_{41} + \dots + v^5 d_{44})}{l_{40}}\right]$$

$$= (16{,}251.47/d)\left((l_{40} - l_{45})/l_{40} - A^{1}_{40:\overline{5}|}\right)$$

$$P = \frac{100{,}000\,A^{1}_{40:\overline{5}|} - (16{,}251.47/d)(1 - {}_5p_{40} - A^{1}_{40:\overline{5}|})}{\ddot{a}_{40:\overline{5}|}}$$

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(23) To evaluate this expression, you need to express $A_{40:\overline{5|}}^{\;\;1}$ in different terms:

$$A_{40:\overline{5|}} = 1 - d\ddot{a}_{40:\overline{5|}} = A_{40:\overline{5|}}^{1} + A_{40:\overline{5|}}^{\;\;\;\;1}$$
$$A_{40:\overline{5|}}^{1} = 1 - d\ddot{a}_{40:\overline{5|}} - A_{40:\overline{5|}}^{\;\;\;\;1}$$
$$= 1 - d\ddot{a}_{40:\overline{5|}} - D_{45}/D_{40}$$

$$D_{40} = N_{40} - N_{41} = 8,452,729 - 7,820,455 = 632,274$$
$$D_{45} = N_{45} - N_{46} = 5,690,850 - 5,245,842 = 445,008$$
$$\ddot{a}_{40:\overline{5|}} = (N_{40} - N_{45})/D_{40} = (8,452,729 - 5,690,850)/632,274$$
$$= 4.3682$$
$$A_{40:\overline{5|}}^{1} = 1 - (.07/1.07)(4.3682) - (445,008/632,274)$$
$$= .01041$$

$$P = \frac{100,000(.01041) - (16251.47(1.07)/.07)(1-(1.07)^5(445,008/632,274) - .01041)}{4.3682}$$
$$= \frac{1,041.06 - 248,415(.002443)}{4.3682}$$
$$= 99.37$$

(A) within "implied range" of 0 to 500

Spring 1988 EA-1A

24 The key to this problem is writing down the present value of benefits (carefully) in terms of probabilities of survival and mortality. The PVB includes both retirement and death benefits.

Death benefits — Retirement benefit

2(40,000) — 2(40,000)(1.04)^1 — 2(40,000)(1.04)^2 — .5(40,000)(1.04)^2

62 — 63 — 64 — 65

$$PVB = 2(40{,}000)(v^1 q_{62}) + 2(40{,}000)(1.04)^1(v^2 p_{62} q_{63}) + 2(40{,}000)(1.04)^2(v^3 p_{62} p_{63} q_{64})$$
$$+ .5(40{,}000)(1.04)^2(v^3 p_{62} p_{63} p_{64})\ddot{a}_{65}^{(12)}$$
$$= \frac{80{,}000(.017)}{1.07} + \frac{80{,}000(1.04)(1-.017)(.0187)}{(1.07)^2} + \frac{80{,}000(1.04)^2(1-.017)(1-.0187)(.0205)}{(1.07)^3}$$
$$+ \frac{20{,}000(1.04)^2(1-.017)(1-.0187)(1-.0205)\,8.74}{(1.07)^3}$$

$$= 1{,}271 + 1{,}336 + 1{,}397 + 145{,}820$$
$$= 149{,}823$$ C

With the varying death benefits, there is no "slick" way to calculate the PVB!

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25 The key to this problem is simplifying the annuity to something you can calculate, and realizing how to use the N_x values to produce the probability of survival.

$$a_{64:65:\overline{2|}} = v\,p_{64}\,p_{65} + v^2\,{}_2p_{64}\,{}_2p_{65}$$
$$= v\,p_{64}\,p_{65} + v^2(p_{64}\,p_{65})(p_{65}\,p_{66})$$
$$= v\,p_{64}\,p_{65}\,[1 + v\,p_{65}\,p_{66}]$$

Age x	N_x	$D_x = N_x - N_{x+1}$	$D_{x+1}/D_x = v\,p_x$
64	294,298	31,254	$.9154 = v\,p_{64}$
65	263,044	28,610	$.9135 = v\,p_{65}$
66	234,434	26,135	$.9113 = v\,p_{66}$
67	208,299	23,818	
68	184,481		

$$v\,p_{64}\,p_{65}\,[1 + v\,p_{65}\,p_{66}]$$
$$= (1+i)\,v\,p_{64}\,(v\,p_{65})\,[1 + (1+i)\,v\,p_{65}\,(v\,p_{66})]$$
$$= 1.07(.9154)(.9135)\,[1 + 1.07(.9135)(.9113)]$$
$$= .8947\,[1 + .8908]$$
$$= 1.6918$$

C