SoftwarePolish

Rick Groszkiewicz
2964 Nestle Creek Drive
Marietta, GA 30062-4857

Voice/fax (770) 971-8913

email: rickg@softwarepolish.com

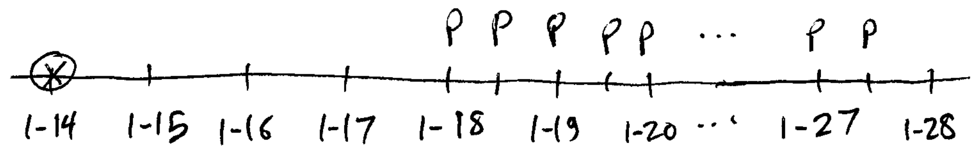
SPRING 2000 EA-1A EXAM SOLUTIONS

THIS PAGE WAS INTENTIONALLY LEFT BLANK

Spring 2000 EA-1A

1

$$1100 = \frac{100}{d} \qquad d = \frac{1}{11} = 1 - v \qquad \Rightarrow \quad i = 10.0\%$$



PV of remaining payments at 1-1-2014 would be 1100 if it included the payment at 1-1-2014. So the present value of the ten year annuity is 1100 − 100 = 1000.

Nominal annual rate ($\frac{1}{2}$ i %) is 5%. Easier to work problem based on semiannual rate, which matches the semiannual payment period:

$$(1+j)^2 = 1.05 \Rightarrow j = 2.47\%$$

$$1000 = (1+j)^{-8} P (\ddot{a}_{\overline{20}|j})$$

$$P = \frac{1000\,(1.05)^4}{\ddot{a}_{\overline{20}|2.47\%}}$$

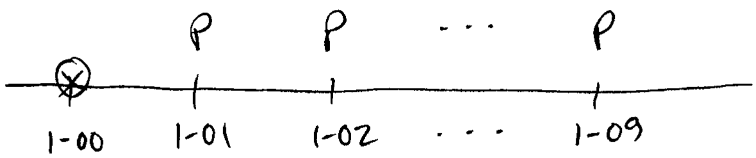
$$= 1000(1.2155)/16.0202$$

$$= 75.87$$

B

Spring 2000 EA-1A

2 There are two ways to work this problem, the easy way and the hard way. For the easy way, you can simply "back into" the value of the death benefit. Start by writing down the mortgage payments:



You can calculate $P = 100{,}000 / a_{\overline{9}|.04}$. The easy way to calculate the PV of the death benefit is to consider the mortgage payments that Smith will make, as long as they survive, equals $P(a_{40:\overline{9}|})$. The PV of the death benefit is the difference between the mortgage at 1-1-2000, and the PV of the expected mortgage payments Smith pays while surviving:

$$\text{PV death benefit} = 100{,}000 - P(a_{40:\overline{9}|})$$

$$= 100{,}000 - \frac{100{,}000}{a_{\overline{9}|.04}}(a_{40:\overline{9}|})$$

$$= 100{,}000 - \frac{100{,}000(7.6923-1)}{7.4353}$$

$$= 100{,}000 - 90{,}007$$

$$= 9{,}993 \qquad \text{E}$$

The harder way to work the problem is to actually write down the series of death benefits and evaluate it. This was shown in 1992 #3, and the result must give the same answer: $P(a_{\overline{9}|.04} - a_{40:\overline{9}|}) = 9{,}993$.

Spring 2000 EA-1A

3 Based on the information given in this problem, you'll have to set up some relationships based on first principles (no A_x or $\ddot{a}_x$ values are given).

One approach to the problem is use of an identity between $\ddot{a}_x$ and A_x, which is $A_x = 1 - d\ddot{a}_x$. Since you are given l_x values, this matches the data in the problem well.

$$A_{73} = 1 - d\ddot{a}_{73} \qquad A_{70} = 1 - d\ddot{a}_{70}$$

$$\ddot{a}_{70} = 1 + v\,p_{70} + v^2\,{}_2p_{70} + v^3\,{}_3p_{70} + \ldots$$

$$= 1 + v\,p_{70} + v^2\,{}_2p_{70} + v^3\,{}_3p_{70}\,(\ddot{a}_{73})$$

You are given the value of A_{70}:

$$10{,}000\,A_{70} = 7{,}000 \Rightarrow A_{70} = .7$$

Once you have a value for A_{73}, then you can solve for the value of $X\%$:

$$7{,}000\,(1 + X/100)^3 = (A_{73})\,10{,}000$$

$$(1 + X/100)^3 = A_{73}/.70$$

$$A_{70} = 1 - d\ddot{a}_{70}$$

$$.7 = 1 - d\left(1 + v\,p_{70} + v^2\,{}_2p_{70} + v^3\,{}_3p_{70}\,(\ddot{a}_{73})\right)$$

$$= 1 - (.06/1.06)\left[1 + \frac{.96}{1.06} + \frac{.90}{(1.06)^2} + \frac{.80}{(1.06)^3}(\ddot{a}_{73})\right]$$

Spring 2000 EA-1A

(3) $.70 = 1 - (.06/1.06)[1 + .9057 + .8010 + .6717(\ddot{a}_{73})]$

$.0566\,[2.7067 + .6717\,\ddot{a}_{73}] = .30$

$$\ddot{a}_{73} = \frac{.30/.0566 - 2.7067}{.6717}$$

$= 3.8609$

Now that you know $\ddot{a}_{73}$, you can solve for A_{73}, and X%

$A_{73} = 1 - d\,\ddot{a}_{73} = 1 - (.06/1.06)3.8609$

$= .7815$

$(1 + X/100)^3 = A_{73}/.70$

$= 1.1164$

$X/100 = 3.74\%$

(E)

An alternative approach is to use solely insurance based identities. This requires almost the same amount of arithmetic and algebraic manipulations:

$A_{70} = A^{1}_{70:\overline{3|}} + v^3\,{}_3p_{70}\,A_{73}$

$A_{73} = (A_{70} - A^{1}_{70:\overline{3|}})(1.06)^3(l_{70}/l_{73})$

$A^{1}_{70:\overline{3|}} = v q_{70} + v^2 p_{70}(q_{71}) + v^3 p_{70}(p_{71}) q_{72}$

$$= v\frac{d_{70}}{l_{70}} + v^2\frac{d_{71}}{l_{70}} + v^3\frac{d_{72}}{l_{70}}$$

$= [(1.06)^{-1}(l_{70} - l_{71}) + (1.06)^{-2}(l_{71} - l_{72}) + (1.06)^{-3}(l_{72} - l_{73})]/l_{70}$

$= [(1.06)^{-1}(4) + (1.06)^{-2}(6) + (1.06)^{-3}(10)]/100$

$= (3.7736 + 5.3400 + 8.3962)/100 = .1751$

Spring 2000 EA-1A

(3) $A^{1}_{70:\overline{3}|} = .1751$

$$A_{73} = (A_{70} - A^{1}_{70:\overline{3}|})(1.06)^3\left(\frac{l_{70}}{l_{73}}\right)$$

$$= (.70 - .1751)(1.1910)\left(\frac{100}{80}\right)$$

$$= .7815$$

The remainder of the solution is identical the previous one:

$$(1 + x/100)^3 = A_{73}/A_{70}$$
$$= 1.1164$$
$$x/100 = 3.74\%$$

Spring 2000 EA-1A

4 This is a common probability question on this exam. The key is knowing the identity for e_x, which is basically a_x using a zero interest:

$$a_x = v p_x + v^2 {}_2p_x + v^3 {}_3p_x + \ldots$$
$$e_x = p_x + {}_2p_x + {}_3p_x + \ldots$$
$$= p_x (1 + e_{x+1})$$

You need the probability of surviving from age 81 to age 83, and you are given the three year probability from age 80 to age 83. You can solve for the value of p_{80}, then back it out to get ${}_2p_{81}$:

$$e_{80} = p_{80}(1+e_{81}) \Rightarrow p_{80} = e_{80}/(1+e_{81})$$
$$= 9.8694/10.3315$$
$$= .9553$$
$${}_3p_{80} = .8642 = p_{80} p_{81} p_{82} = .9553(p_{81} p_{82})$$
$${}_2p_{81} = p_{81} p_{82} = .8642/.9553$$
$$= .9047$$

The last part is knowing some probability concepts. The probability that at least two die before age 83 equals the sum of (i) probability exactly two die (3 cases) plus (ii) probability exactly three die. There are 3 cases for item (i), since any one of the three could be the one who survives to age 83.

Spring 2000 EA-1A

(4) $Pr(\text{At least 2 die}) = Pr(\text{Exactly 2 die}) + Pr(\text{Exactly 3 die})$

$$= 3({}_2p_{81})(1-{}_2p_{81})^2 + (1-{}_2p_{81})^3$$

$$= 3(.9047)(1-.9047)^2 + (1-.9047)^3$$

$$= .0247 + .0009$$

$$= .0255$$

D

The other way to express the probability is this

$Pr(\text{At least 2 die}) = 1 - Pr(\text{None dies}) - Pr(\text{Exactly 1 dies})$

$$= 1 - ({}_2p_{81})^3 - 3(1-{}_2p_{81})({}_2p_{81})^2$$

$$= 1 - (.9047)^3 - 3(1-.9047)(.9047)^2$$

$$= 1 - .7404 - .2341$$

$$= .0255$$

Again, there are three cases to consider when exactly one person dies, since any one of the three could be the one who died prior to age 83.

Spring 2000 EA-1A

5 The key to working this problem is using Makeham's law, which simplifies the calculations for multiple bonds quite a bit. You can get the same answer with any other bond price formula, but it may take longer.

Makeham's formula: $P = K + (g/i)(C-K)$
$$= Cv^n + \frac{Fr}{Ci}(C - Cv^n)$$

For this problem, you need to calculate the price for the set of 20 bonds to produce the yield rate of 2.5% semiannually. Then you can look at the price of any one bond to determine its individual yield rate.

All bonds at 1-1-2000: $\Sigma P = \Sigma\left(Cv^n + Fr/Ci\right)\left(C - Cv^n\right)\right)$
$$= \Sigma\left(K + \frac{5000(.020)}{5000(.025)}(5000-K)\right)$$
$$= \Sigma\left[K + .8(5000-K)\right]$$
$$= \Sigma(4000 + .2K)$$
$$= 20(4000) + .2\,\Sigma(5000v^n)$$

Since the redemption dates are annual, you should determine the annual effective rate to calculate the price:

$(1.025)^2 - 1 = 5.06\%$ per annum

Spring 2000 EA-1A

(5) All 20 bonds - $\underset{1\text{-}1\text{-}00}{\Sigma \text{Prices}} = 20(4000) + .2(5000)\left(\underbrace{v^{\overset{1\text{-}1\text{-}15}{15}} + v^{16} + \dots + v^{34}}_{20 \text{ redemption dates}}\right)$ at 5.06%

$= 80{,}000 + 1{,}000\left(\ddot{a}_{\overline{35}|} - \ddot{a}_{\overline{15}|5.06\%}\right)$

$= 86{,}209$

Since each bond has the same price, it is $\frac{1}{20}(86{,}209) = 4{,}310$.
On an individual basis, the yield rates will vary for each bond. I'll assume the yield rate is 5.06% per annum, and solve for the redemption date (actually the # of years)

For one bond at 1-1-00 $\quad \text{Price} = 4000 + .2(5000)v^n$

$4310 = 4000 + 1000(1.0506)^{-n}$

$.310 = (1.0506)^{-n}$

$n = \dfrac{-\log(.310)}{\log(1.0506)}$

$= 23.69$

bonds with earlier redemption dates have highest yield rates, so those redeemed after 15, 16, ..., 22 and 23 years earn more than 5.06%. Thus nine of the 20 bonds have a yield rate greater than 5.06% $\Rightarrow 9/20 = 45\%$

B

Spring 2000 EA-1A

6 The key to working this problem is the ability to interpret the information given into probability relationships. Since Smith and Brown receive 6,000 per annum, the problem tells you that the probability is based on Smith dying within 2 years, and Brown dying within 3 years.

Since both Smith and Brown are age 80 at 1/1/2000, you will use the individual probabilities of death to construct the combination that also requires Brown to die after Smith:

Age 80 81 82 83

Smith must die at 80 or 81 — Brown must die at 80, 81, or 82

Brown must die after Smith

Case 1: Smith dies at age 80

Brown must die at ages 80, 81 or 82. One typical trick is that they both may die in the same year of age, and you must allow for the 50% chance that Brown may NOT die after Smith:

$$\text{Probability} = \underbrace{q_{80}}_{\text{Smith Dies}} \underbrace{\left[q_{80}(.50) + p_{80}q_{81} + p_{80}p_{81}q_{82} \right]}_{\text{Brown Dies at ages 80, 81 or 82}}$$

$$= .0813[.0813(.50) + .9187(.0885) + .9187(.9115)(.0962)]$$

$$= .0813[.0407 + .0813 + .0806]$$

$$= .0165$$

(6) <u>Case 2: Smith dies at age 81</u>

Now both must survive age 80, and Brown must die at age 81 or 82. You must allow for the fact that half of the time, the order of death may be incorrect when both Smith and Brown die at age 81:

$$\text{Probability} = \underbrace{p_{80}q_{81}}_{\text{Smith Dies at age 81}}\,\underbrace{[\,p_{80}q_{81}(.50) + p_{80}p_{81}q_{82}\,]}_{\text{Brown Dies at ages 81 or 82}}$$

$$= .9187(.0885)[\,.9187(.0885)(.50) + .9187(.9115)(.0962)\,]$$

$$= .0813\,[\,.0407 + .0806\,]$$

$$= .0099$$

$$\text{Total probability} = .0165 + .0099$$

$$= .0263$$

C

Comparing the formulas for the probabilites in both cases, there are several common terms. You could have reduced the arithmetic by combining terms

$$q_{80}\,[\,q_{80}(.5) + p_{80}q_{81} + p_{80}p_{81}q_{82}\,]$$

$$+\, p_{80}q_{81}\,[\qquad p_{80}q_{81}(.5) + p_{80}p_{81}q_{82}\,]$$

$$= (q_{80})^2(.50) + (p_{80}q_{81})(q_{80} + p_{80}q_{81}(.5)) + (p_{80}p_{81}q_{82})(q_{80} + p_{80}q_{81})$$

$$= (.0813)^2(.5) + .9187(.0885)[.0813 + .9187(.0885).5] + .9187(.9115)(.0962)(.0813 + .9187(.0885))$$

$$= .0033 + .0813(.1220) + .0806(.1626)$$

$$.0263 = .0033 + .0099 + .0131$$

Spring 2000 EA-1A

7 The key to this problem is writing down each loan payment carefully, since the amount changes each year:

Year	Pay	Loan pmt
2000	3000/mo	60/mo = 2% (3,000/mo)
2001	3,150/mo	63/mo = 2% (3,000/mo)(1.05)

On a time diagram, the loan payments look like this

	60	....	60	63	....	63
1-1-00	1-31	....	12-31-00	1-31-01	...	12-31-01

The question asks for the outstanding loan balance at 1-1-02. There are usually two methods of solution

Prospective O/S Balance = PV of O/S loan pmts

Retrospective O/S Balance = Accum loan − Accum prior pmts

In order to avoid figuring out the total number of future loan payments, it will be simpler to use the retrospective approach in this problem

1-1-02 O/S balance $= 5{,}000(1.01)^{24} - 60(s_{\overline{12}|.01})(1.01)^{12} - 63(s_{\overline{12}|.01})$

$= 5{,}000(1.01)^{24} - s_{\overline{12}|.01}\left[60(1.01)^{12} + 63\right]$

$= 5{,}000(1.2697) - 12.6825\left[60(1.1268) + 63\right]$

$= 4{,}692$

C

Spring 2000 EA-1A

8 modified duration equals the regular duration divided by $(1+i)$. Regular duration is a weighted average of the time at payment, where the weight equals the present value of each payment:

$$\text{modified duration} = \frac{\bar{d}}{1+i} \qquad \bar{d} = \frac{\Sigma\, t\,(v^t R_t)}{\Sigma\,(v^t R_t)}$$

The time diagram is easy for the payments under this bond:

$$\begin{array}{cccccc} & 70 & 70 & \cdots & 70 & \begin{matrix}1000\\70\end{matrix} \\ \hline 0 & 1 & 2 & \cdots & 19 & 20 \end{array}$$

with 7% annual coupons on a 1,000 bond, there will both be a coupon of 70 and the 1,000 redemption after 20 years. You are told that the force of interest is .07, which allows you to calculate the yield rate

$$e^{\delta} = 1+i = 1.0725$$

$$\text{modified duration} = y = (1.0725)^{-1}\left[\frac{70(1\cdot v + 2\cdot v^2 + \ldots + 20\cdot v^{20}) + 1{,}000\,(20\cdot v^{20})}{70(v + v^2 + \ldots + v^{20}) + 1{,}000\,(v^{20})}\right]$$

$$= (1.0725)^{-1}\left[\frac{70(Ia_{\overline{20|}.0725}) + 20{,}000\,v^{20}}{70(a_{\overline{20|}.0725}) + 1{,}000\,v^{20}}\right]$$

$$= (1.0725)^{-1}\left[\frac{70(\ddot{a}_{\overline{20|}.0725} - 20v^{20})/.0725 + 20{,}000\,v^{20}}{70(\ddot{a}_{\overline{20|}.0725})/1.0725 + 1{,}000\,v^{20}}\right]$$

$$= (1.0725)^{-1}\left[\frac{70(6.2121)/.0725 + 4{,}932}{70(11.144)/1.0725 + 247}\right]$$

$$= 10.4629$$

A

Spring 2000 EA-1A

9 This is a straightforward premium calculation problem. The insurance is 30 year term life, and you should assume premium payments over the same 30 year period:

$$P = 100{,}000 \frac{A^{1}_{35:\overline{30|}}}{\ddot{a}_{35:\overline{30|}}} = 100{,}000 \frac{(M_{35}-M_{65})/D_{35}}{(N_{35}-N_{65})/D_{35}}$$

The key to working the problem is knowing the various identities between M_x, D_x and N_x:

$M_x = D_x - d \cdot N_x$ or $M_x = v \cdot N_x - N_{x+1}$

You can use either of these to determine the premium.

$$P = 100{,}000 \left(\frac{M_{35}-M_{65}}{N_{35}-N_{65}} \right)$$

$$= 100{,}000 \left(\frac{vN_{35}-N_{36}-(vN_{65}-N_{66})}{N_{35}-N_{65}} \right)$$

$$= 100{,}000 \left(\frac{v(N_{35}-N_{65})-(N_{36}-N_{66})}{N_{35}-N_{65}} \right)$$

$$= 100{,}000 \left(v - \frac{N_{36}-N_{66}}{N_{35}-N_{65}} \right)$$

$$= 100{,}000 \left(\frac{1}{1.07} - \frac{11{,}470{,}471-773{,}639}{12{,}364{,}661-868{,}053} \right)$$

$$= 100{,}000 \,(.9346 - .9304)$$

$$= 414.57$$

A

Spring 2000 EA-1A

10 The key to working this problem is simply writing down the correct time diagram for the payments:

300	600	...	3,000	2,800	2,600	...	1,000
1-00	1-01	...	1-09	1-10	1-11	...	1-19

$$\underbrace{\text{1-00} \quad \text{1-01} \quad \cdots \quad \text{1-09}}_{\substack{\text{10 payments} \\ \text{7\% for 1st 10 years}}} \qquad \underbrace{\text{1-10} \quad \text{1-11} \quad \cdots \quad \text{1-19}}_{\substack{\text{10 payments} \\ \text{6\% for 2nd 10 years}}}$$

The easiest way to work this problem is knowing the formula for an increasing annuity: $(Ia)_{\overline{n}|i} = \dfrac{\ddot{a}_{\overline{n}|i} - nv^n}{i}$.

Even if you did not know the formula, you could write out the present value "long hand", and then derive its value through simple algebraic manipulations.

$$PV = 300(I\ddot{a}_{\overline{10}|.07}) + (1.07)^{-10}\left[2800\,\ddot{a}_{\overline{10}|.06} - 200\,(Ia)_{\overline{9}|.06}\right]$$

You can verify the payment amounts using this formula:
$300(10) = 3000$ at 1-1-09, and $2800 - 200(9) = 1{,}000$ at 1-1-19.

$$\begin{aligned}
PV &= 300(1.07)\left(\ddot{a}_{\overline{10}|.07} - 10(1.07)^{-10}\right)/.07 + (1.07)^{-10}\left(2800\,\ddot{a}_{\overline{10}|.06} - 200\left[\ddot{a}_{\overline{9}|.06} - 9(1.06)^{-9}\right]/.06\right) \\
&= 321(7.5152 - 10(.5083))/.07 + .5083\left(2800(7.8017) - 200\left[7.2098 - 9(.5919)\right]/.06\right) \\
&= 321(34.739) + .5083(21{,}845 - 6{,}276) \\
&= 11{,}151 + 7{,}915 \\
&= 19{,}066
\end{aligned}$$

D

Spring 2000 EA-1A

(10) If you don't know the formula for an increasing annuity, you can still work the problem. You'll use algebraic manipulations to simplify the expression for the present value of the series of payments.

Since you have two different interest rates, it will be easier if you separately determine the present values of the first ten and last ten payments. For the first 10 you essentially derive the increasing annuity formula:

$$Q = 300 + 600v + \dots + 2700v^8 + 3000v^9 \quad \text{all at } 7\%$$

$$Qv = 300v + \dots + 2400v^8 + 2700v^9 + 3000v^{10}$$

$$Q(1-v) = 300 + 300v + \dots + 300v^8 + 300v^9 - 3000v^{10}$$

$$Q = \frac{300\,\ddot{a}_{\overline{10}|.07} - 3000(1.07)^{-10}}{1-v} = 11{,}151$$

The last ten payments are slightly more complicated

$$P = (1.07)^{-10}\left[2800 + 2800\,a_{\overline{9}|.06} - 200(1.06)^{-1} - 400(1.06)^{-2} - \dots - 1{,}600(1.06)^{-8} - 1{,}800(1.06)^{-9}\right]$$

$$P(1.06) = (1.07)^{-10}\left[2800(1.06 + \ddot{a}_{\overline{9}|.06}) - 200 - 400(1.06)^{-1} - 600(1.06)^{-2} - \dots - 1800(1.06)^{-8}\right]$$

$$.06P = (1.07)^{-10}\left[2800(.06 + \ddot{a}_{\overline{9}|} - a_{\overline{9}|.06}) - 200(1 + (1.06)^{-1} + \dots (1.06)^{-8}) + 1800(1.06)^{-9}\right]$$

$$= (1.07)^{-10}\left[.06(2800)\,\ddot{a}_{\overline{10}|.06} - 200\,\ddot{a}_{\overline{9}|.06} + 1800(1.06)^{-9}\right]$$

$$P = \frac{.5083}{.06}\left[.06(2800)\,7.8017 - 200(7.2098) + 1800(.5919)\right)$$

$$= 8.4725\left[1{,}311 - 1{,}442 + 1{,}065\right)$$

$$= 7{,}915$$

Total $PV = P + Q = 11{,}151 + 7{,}915 = 19{,}066$ as expected

Spring 2000 EA-1A

Solution revised 05/02/02

11 This is the first time this particular multiple decrement problem was asked on the exam. Most earlier problems are based on the assumption of uniform distribution of decrements in the multiple decrement table. This produces formulas such as Jordan 14.316 $q'^{(1)}_x \doteq \frac{q^{(1)}_x}{1-\frac{1}{2}q^{(2)}_x} = \frac{q^{(1)}_x}{1-\frac{1}{2}(q^{(T)}_x - q^{(1)}_x)}$

This problem uses a different assumption, which is uniform distribution of decrements in the single decrement tables. For a two decrement table, the resulting formula is $q^{(1)}_x \doteq q'^{(1)}_x\left[1-\frac{1}{2}q'^{(2)}_x\right]$, which

can be rewritten as $q'^{(1)}_x \doteq \frac{q^{(1)}_x}{1-\frac{1}{2}q'^{(2)}_x}$

Compared to the earlier formula, we are using the absolute rate of decrement in the denominator instead of the probability. You were supposed to intuitively determine this on the 2000 exam! A detailed proof for the formula follows the solution of this problem.

$\frac{d^{(w)}_{55}}{l^{(T)}_{55}} = q^{(w)}_{55} = q'^{(w)}_{55}\left[1-\frac{1}{2}q'^{(d)}_{55}\right]$ based on given formula

$$\therefore d^{(w)}_{55} = l^{(T)}_{55}\, q'^{(w)}_{55}\left[1-.5\,q'^{(d)}_{55}\right]$$
$$= 7120(.040747)[1-.5(.009033)]$$
$$= 288.81$$

B

Spring 2000 EA-1A Solution revised 05/02/02

(11) The derivation of the formula based on U.D.D. in the *single* decrement tables is shown in Actuarial Mathematics §10.6. Formula 10.6.3 is derived based on a 3 decrement table; the following derivation is quite similar:

10.2.12 $\mu_x^{(j)}(t) = \dfrac{1}{{}_tp_x^{(T)}} \cdot \dfrac{d}{dt}\,{}_tq_x^{(j)}$

10.5.3 $\int_0^1 {}_tp_x^{(T)}\,\mu_x^{(j)}(t)\,dt = q_x^{(j)}$

${}_tp_x^{(T)} = {}_tp_x'^{(1)}\,{}_tp_x'^{(2)}$ for a double decrement table

${}_tp_x'^{(j)} = 1 - {}_tq_x'^{(j)} = 1 - t \cdot q_x'^{(j)}$ assuming U.D.D.

$q_x^{(1)} = \int_0^1 {}_tp_x'^{(1)}\,{}_tp_x'^{(2)}\,\mu_x^{(1)}(t)\,dt$ from 10.5.3

$= \int_0^1 \left[{}_tp_x'^{(1)}\,\mu_x^{(1)}(t)\right]\left(1 - t \cdot q_x'^{(2)}\right) \cdot dt$

$= q_x'^{(1)} \int_0^1 \left(1 - t \cdot q_x'^{(2)}\right) dt$

$= q_x'^{(1)} \left[t - \dfrac{t^2}{2} q_x'^{(2)} \right]_0^1$

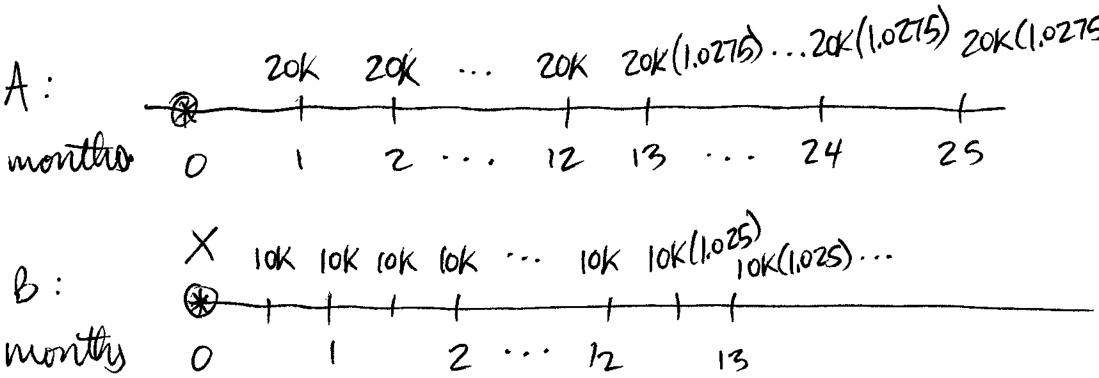
$= q_x'^{(1)} \left[1 - \tfrac{1}{2} q_x'^{(2)} \right]$

My original solution for this problem incorrectly used various formulas based on assumption of U.D.D. in the multiple decrement tables. It was unfortunate that they gave very similar answers!

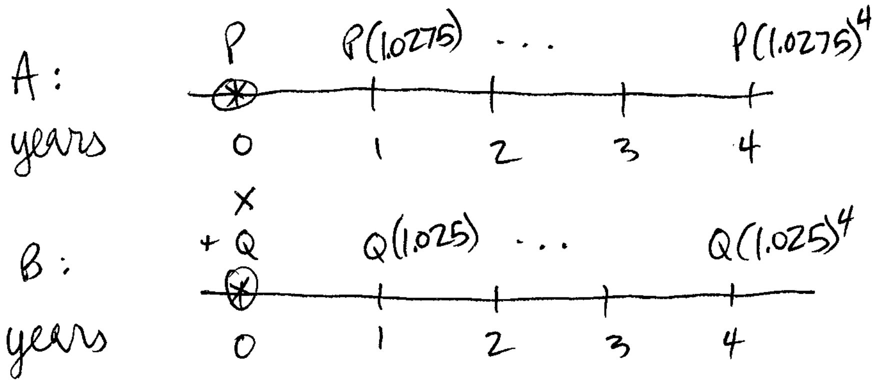
Spring 2000 EA-1A

12 This problem sets a record for the number of different interest rates that you must derive. The key to the problem is recognizing that the present values of the contracts are easier to determine based on annual payments, which correspond to the annual increases in the monthly (or semimonthly) payments.

Here are the time diagrams for the payments under both contracts (all payments for 5 years are not shown):



The goal is to replace these payments with annual ones, which makes the present values simpler to calculate:



Spring 2000 EA-1A

(12) For contract A, you need to determine the monthly rate of interest to determine the annual payment P:

$P = 20{,}000\, a_{\overline{12}|i} \qquad 1+i = \left(1+\frac{.06}{24}\right)^2 = (1.0025)^2 = 1.00501$

$= 20{,}000(\ddot{a}_{\overline{12}|.501\%})/(1.00501)$

$= 232{,}369$

$PV{:}A = P + P\left(\frac{1.0275}{1+j}\right) + \dots + P\left(\frac{1.0275}{1+j}\right)^4$

The annual interest rate is j, equivalent to the given semimonthly rate of .06/24 = .25%:

$1+j = (1+.06/24)^{24} = 1.06176$

To calculate the present value of contract A, you have

$PV{:}A = P\,\ddot{a}_{\overline{5}|k}$ where $1+k = \frac{1.06176}{1.0275} = 1.03334$

$= 232{,}369\, \ddot{a}_{\overline{5}|3.334\%}$

$= 1{,}089{,}254$

Now, do similar steps for Contract B, but don't forget X!

$Q = 10{,}000\, a_{\overline{24}|.25\%}$

$= 10{,}000(\ddot{a}_{\overline{24}|.25\%})/(1.0025)$

$= 232{,}660$

$PV{:}B = Q + Q\left(\frac{1.025}{1+j}\right) + \dots + Q\left(\frac{1.025}{1+j}\right)^4 + X$

$= Q\,\ddot{a}_{\overline{5}|l} + X$ where $1+l = \frac{1.06176}{1.0250} = 1.03586$

$= X + 232{,}660\, \ddot{a}_{\overline{5}|3.586\%}$

$= X + 1{,}085{,}495$

Now, set the two present values equal to solve for X:

$1{,}089{,}254 = X + 1{,}085{,}495 \Rightarrow X = 3{,}759$ (B)

Spring 2000 EA-1A

13 The key to working this problem is knowing how to write down the outstanding loan balance at any point in time. There are two formulas you could use

Prospective O/S loan balance = PV of future loan payments

Retrospective O/S loan balance = Accumulated original O/S loan − accumulated past loan payments

In this problem, you should use the prospective formula. You can determine the original loan payment at 1-1-1990

$$100{,}000 = P\, a_{\overline{360}|.01}$$

$$P = (100{,}000 / 97.2183) = 1028.61$$

After 120 payments, the outstanding loan balance equals the PV of the remaining 240 payments $= P\, a_{\overline{240}|.01}$. You are told that the payments in the future will be increased by Q, and that there will be a total of 160 payments. This means there will only be 40 future payments of P+Q:

$$P\, a_{\overline{240}|.01} = (P+Q)\, a_{\overline{40}|.01}$$

$$P+Q = P\,(a_{\overline{240}|.01} / a_{\overline{40}|.01})$$

$$Q = P\,(a_{\overline{240}} / a_{\overline{40}}) - P$$

$$= 1028.61\,(90.8194/32.8347) - 1028.61$$

$$= 2845 - 1029$$

$$= 1816$$

D

Spring 2000 EA-1A

14 The key to working this problem is recognizing that the annual increase in the death benefit is the same as the ratio of the d_x values. You should write out the death benefits on a time line, and then write down the present value of the death benefits:

$$\begin{array}{cccccc} 10{,}000 & 10{,}000(1.11)^1 & \dots & & (10{,}000)(1.11)^4 & \\ \hline 75 & 76 & 77 & 78 & 79 & \end{array}$$

$$PV = 10{,}000\, v\, q_{75} + 10{,}000(1.11)^1 v^2\, {}_{1|}q_{75} + \dots + 10{,}000(1.11)^4 v^5\, {}_{4|}q_{75}$$

You should write this in terms of d_x and l_x values to use the information given in the problem.

$$PV = \frac{10{,}000}{1.07}\left(\frac{d_{75}}{l_{75}}\right) + \frac{10{,}000(1.11)}{(1.07)^2}\left(\frac{d_{76}}{l_{75}}\right) + \dots + \frac{10{,}000(1.11)^4}{(1.07)^5}\left(\frac{d_{79}}{l_{75}}\right)$$

$$= \frac{10{,}000}{1.07}\left[\frac{d_{75}}{l_{75}} + \frac{1.11}{1.07}\left(\frac{d_{76}}{l_{75}}\right) + \dots + \left(\frac{1.11}{1.07}\right)^4 \frac{d_{79}}{l_{75}}\right]$$

You are told that $(d_{75}/d_{76}) = 1.11 \Rightarrow d_{76} = (d_{75}/1.11)$

$(d_{76}/d_{77}) = 1.11 \Rightarrow d_{77} = d_{75}/(1.11)^2$

$$PV = \frac{10{,}000}{1.07}\left[\frac{d_{75}}{l_{75}} + \frac{d_{75}}{(1.07)l_{75}} + \dots + \frac{d_{75}}{(1.07)^4 l_{75}}\right]$$

$= (10{,}000/1.07)(d_{75}/l_{75})(1 + v + \dots + v^4)$

$= (10{,}000/1.07)(.10)\,\ddot{a}_{\overline{5}|.07}$

$= 9346(.10)(4.3872)$

$= 4100$ (B)

Spring 2000 EA-1A

15 This problem has never been asked on the exam before. To work it, you must know the formula for the probability of survival based on central rates of decrement:

Jordan 14.15 $\quad m_x^{(k)} \doteq \dfrac{q_x^{(k)}}{1-\frac{1}{2}q_x^{(T)}}$ $\quad$ central rate of decrement

Jordan 14.17 $\quad p_x^{(T)} \doteq \dfrac{1-\frac{1}{2}m_x^{(T)}}{1+\frac{1}{2}m_x^{(T)}}$ $\quad$ where $m_x^{(T)} = \Sigma m_x^{(k)}$

Since the problem gives you the central rates of decrement, its not much work to plug in the values:

$$m_x^{(T)} = .02 + .10 = .12$$

$$p_x^{(T)} = \frac{1-\frac{1}{2}(.12)}{1+\frac{1}{2}(.12)}$$
$$= (1-.06)/(1+.06)$$
$$= .94/1.06$$
$$= .8868$$

Number of survivors after one year is 10,000(.8868) = 8,868

C

Spring 2000 EA-1A

16 The key to working this problem is knowing what the survival function is. It gives the probability of survival from age zero to age x:

$p_x = l_{x+1}/l_x$

$l_x = l_0 [S(x)]$

You need to construct the probability asked for in terms of l_x values, then plug in the given $S(x)$ function.

Probability = 1 − [Probability (survive to age 51)][Probability (die between age 51 and age 64)]

$$= 1 - \frac{l_{51}}{l_{19}}\left(\frac{l_{51}-l_{64}}{l_{51}}\right)$$

$$= 1 - \frac{l_{51}-l_{64}}{l_{19}}$$

Now you need to calculate the values of l_{19}, l_{51}, and l_{64}:

$l_{51} = l_0 [\sqrt{100-51}/10] = l_0(\sqrt{49}/10) = l_0(.7)$

$l_{64} = l_0 [\sqrt{100-64}/10] = l_0(\sqrt{36}/10) = l_0(.6)$

$l_{19} = l_0 [\sqrt{100-19}/10] = l_0(\sqrt{81}/10) = l_0(.9)$

$$\therefore \text{Probability} = 1 - \frac{l_0(.7) - l_0(.6)}{l_0(.9)}$$

$$= 1 - \frac{(.1)}{(.9)} = \frac{8}{9} = .8889$$

C

Spring 2000 EA-1A

17 The time and dollar weighted return problems always require you to write the market values and cash flows on a time line diagram:

	1/1	4/1	7/1	10/1	12/31
MV before		60,000	45,000	40,000	65,000
Cash flows		0-P	17,000-P	55,000-P	0
MV after	50,000	60,000-P	62,000-P	95,000-P	65,000

Based on the dates, you are given the market values prior to each cash flow. Then you simply add the cash flow and that gives the market value after the cash flow.

DOLLAR WEIGHTED

You are told the dollar weighted return is 7.00%. You need to use this to derive the value of P. You should write down the formula for the dollar weighted return using simple interest, as specified in the problem:

$$50{,}000\left(1+\tfrac{4}{4}i\right) - P\left(1+\tfrac{3}{4}i\right) + (17{,}000-P)\left(1+\tfrac{2}{4}i\right) + (55{,}000-P)\left(1+\tfrac{1}{4}i\right) = 65{,}000$$

$$65{,}000 = \left[50{,}000 - P + (17{,}000-P) + (55{,}000-P)\right] + \left[50{,}000 - .75P + (8{,}500 - .5P) + (13{,}750 - .25P)\right]i$$

$$= 122{,}000 - 3P + \left[72{,}250 - 1.5P\right](.0700)$$

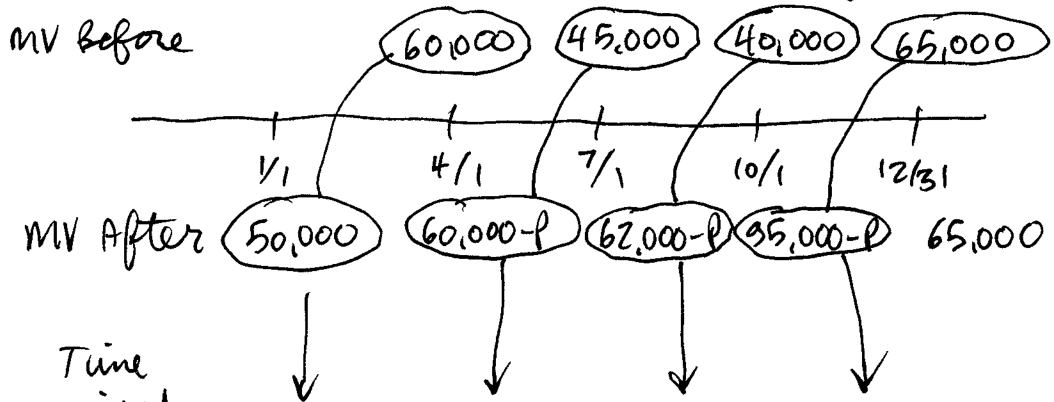
$$3.105P = 122{,}000 + 5057.5 - 65{,}000$$

$$P = 19{,}986.3$$

Spring 2000 EA-1A

(17) TIME WEIGHTED

Now that you have the value of P, you can write down the formula for the time weighted return. This is done based on the previous time line:



Time weighted

$$1+j = \left(\frac{60{,}000}{50{,}000}\right)\left(\frac{45{,}000}{60{,}000-P}\right)\left(\frac{40{,}000}{62{,}000-P}\right)\left(\frac{65{,}000}{95{,}000-P}\right)$$

It is convenient that the numerators are all at the top of the diagram, and the denominators are at the bottom. Now plug in the value of P, and solve for j:

$$1+j = \frac{60{,}000}{50{,}000}\left(\frac{45{,}000}{40{,}014}\right)\left(\frac{40{,}000}{42{,}014}\right)\left(\frac{65{,}000}{75{,}014}\right)$$

$$= (1.2)(1.1246)(.9521)(.8665)$$

$$= 1.1133$$

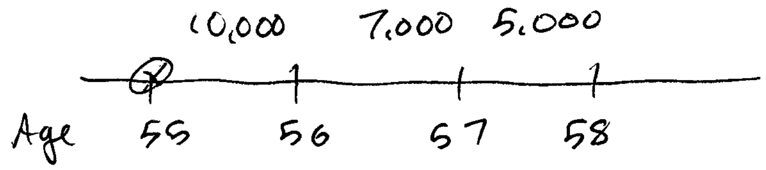
$$j = 11.33\%$$

D

The time weighted return always uses ratios of the market values taken at the date each cash flow occurs, and measures the change in market value between cash flows.

Spring 2000 EA-1A

18 In this problem, you should write down the death benefits on a time line diagram:



You have a varying insurance for three years. You can write down the single premium based on "first principles"

$$NSP = 10{,}000\, v^1 q_{55} + 7{,}000\, v^2 p_{55} q_{56} + 5{,}000\, v^3 p_{55} p_{56} q_{57}$$

You need to know how to derive the p_x and q_x values based on the given information. The key to the problem is knowing this formula for e_x:

$$a_x = v p_x + v^2 {}_2p_x + \cdots = v p_x (1 + a_{x+1})$$

$$e_x = p_x + {}_2p_x + \cdots = p_x (1 + e_{x+1})$$

$$p_x = e_x / (1 + e_{x+1})$$

$$p_{55} = e_{55} / (1 + e_{56}) = 22/22.20 = .9910 \qquad q_{55} = .0090$$

$$p_{56} = e_{56} / (1 + e_{57}) = 21.2/21.4 = .9907 \qquad q_{56} = .0093$$

$$p_{57} = e_{57} / (1 + e_{58}) = 20.4/20.6 = .9903 \qquad q_{57} = .0097$$

$$NSP = 10{,}000 (1.07)^{-1} (.0090) + 7{,}000 (1.07)^{-2} (.9910)(.0093) + 5{,}000 (1.07)^{-3} (.9910)(.9907)(.0097)$$

$$= 84.20 + 56.63 + 38.90$$

$$= 179.72$$

A

Spring 2000 EA-1A

19 This is a stationary population problem that is similar to 1992 #6. The key is knowing the formula for the average age at termination for everyone who terminates after age x:

$$\text{Average age at termination} = x + \frac{T_x - T_y - (y-x)\,l_y}{l_x - l_y}$$

Based on the data given in the problem, $x = 30$, $y = 60$, $T_{30} - T_{60} = 5000$ lives in the stationary population, and $l_{30} - l_{60} = 100$ terminations each year.

The problem asks for the number of new members in the stationary population each year, which is l_{30}. Based on the information given, you can solve for the value of l_{60}, and then use that to determine l_{30}:

$$35 = 30 + \frac{T_{30} - T_{60} - 30\,l_{60}}{l_{30} - l_{60}}$$

$$= 30 + (5000 - 30\,l_{60})/100$$

$$500 = 5000 - 30\,l_{60}$$

$$l_{60} = 150$$

$$\therefore\ l_{30} = 100 + l_{60}$$

$$= 250$$

C

Spring 2000 EA-1A

20 Force of mortality problems are infrequent on the exam, and this problem is similar to the first multiple decrement force of mortality problem, which was 1996 #1.

There are two formulas you can use to work the problem:

Jordan 14.22 $\quad \mu_x^{(k)} = -\dfrac{1}{l_x^{(T)}}\left[\dfrac{d\, l_x^{(k)}}{dx}\right]$

Jordan 14.26 $\quad \mu_x^{(k)} \doteq \dfrac{d_{x-1}^{(k)} + d_x^{(k)}}{2\, l_x^{(T)}}$

The easiest calculation uses the approximation formula

$$\mu_{40}^{(1)} \doteq \frac{d_{39}^{(1)} + d_{40}^{(1)}}{2\, l_{40}^{(T)}} \doteq \frac{(e^{-39} - e^{-40}) + (e^{-40} - e^{-41})}{2(100-40)\, e^{-40}}$$

$$\doteq \frac{e^{1} - e^{-1}}{2(60)}$$

$$\doteq 2.35/120 \doteq .0196$$

NOTE - this is only an approximation

D

You can use the exact formula, but you need to derive the formula for $l_x^{(1)}$ to get the derivative with respect to x:

$$d_x^{(1)} = e^{-x} - e^{-x-1}$$

$$l_x^{(1)} = \sum_{t=x}^{w} d_t^{(1)} = (e^{-x} - e^{-x-1}) + (e^{-x-1} - e^{-x-2}) + \ldots + (e^{-w} - e^{-w-1})$$

Based on the formula for $l_x^{(T)}$, it appears that $w = 100$, since $l_{100}^{(T)} = 0$.

$$\therefore l_x^{(1)} = e^{-x} + e^{-100}$$

$$\mu_x^{(1)} = \frac{-1}{l_x^{(T)}}\left[\frac{d\, l_x^{(1)}}{dx}\right] = \left(\frac{-e^{x}}{100-x}\right)\left(-e^{-x}\right) = \frac{1}{100-x}$$

Exact value $\therefore \mu_{40}^{(1)} = 1/(100-40) = 1/60 = .0167$ also in range D

Spring 2000 EA-1A

21 The key to working gross premium problems is writing down each expense item carefully. You also need to construct the annuities so that you can evaluate them based on the few commutation functions given.

In general, you'll equate the present value of the gross premium to the present value of expenses and death benefits:

$$G \cdot \ddot{a}_{60} = 10{,}000\, A_{60} + 5(10)\, A_{60} + .75G$$
$$+ .20G(\text{years } 2\text{-}10) + .05G(\text{years } 11+) + 10\, \ddot{a}_{60}$$

The best way to express the varying layers of gross premium expenses is based on annuities due starting at age 60, which requires you to think of .75G as .55 + .15 + .05 ; you also should consider the .20G as .15 + .05 :

$$G \cdot \ddot{a}_{60} = 10{,}050 A_{60} + .55G + .15G\, \ddot{a}_{60:\overline{10}|} + .05G\, \ddot{a}_{60} + 10\, \ddot{a}_{60}$$

$$A_{60} = 1 - d\ddot{a}_{60} \qquad D_{60} = N_{60} - N_{61} \qquad \ddot{a}_{60} = N_{60}/D_{60}$$

$$G = \frac{10{,}050(1 - d\ddot{a}_{60}) + 10\, \ddot{a}_{60}}{\ddot{a}_{60} - .55 - .15(\ddot{a}_{60} - N_{70}/D_{60}) - .05\ddot{a}_{60}}$$

$$= \frac{10{,}050(1/\ddot{a}_{60} - d) + 10}{.95 - .55/\ddot{a}_{60} - .15(1 - N_{70}/N_{60})}$$

$$= \frac{10{,}050[(36{,}709/352{,}087) - .08/1.08] + 10}{.95 - .55(36{,}709/352{,}087) - .15(1 - 108{,}886/352{,}087)}$$

$$= 313.38/.7869$$

$$= 398.24$$

B

Spring 2000 EA-1A

22 This problem is mostly one of interpreting what you are given. Assume that half the loan is paid back by typical 7% interest amortization over 10 years:

$$\text{Annual pmt}_1 = \frac{50\%(x)}{a_{\overline{10}|.07}}$$

The other half of x is repaid via a sinking fund. This means that, at the end of 10 years, the sinking fund will accumulate to 50%(x) based on the 6% rate. In addition, the interest of .07(50% x) is paid out of pocket yearly:

$$\text{Annual pmt}_2 = .07(50\%(x)) + \frac{50\% x}{s_{\overline{10}|.06}}$$

The sum of the two payments is equal to 1000:

$$1000 = \frac{.5x}{a_{\overline{10}|.07}} + .07(.5x) + \frac{.5x}{s_{\overline{10}|.06}}$$

$$= .5x\left(\frac{1}{a_{\overline{10}|.07}} + .07 + \frac{1}{s_{\overline{10}|.06}}\right)$$

$$x = 2000 \Big/ \left(\frac{1}{a_{\overline{10}|.07}} + .07 + \frac{1}{s_{\overline{10}|.06}}\right)$$

$$= 2000/.288$$

$$= 6939$$

B

Spring 2000 EA-1A

23 This is one of the rare questions on Gompertz' law.
For a mortality table subject to Gompertz' law, you can replace joint life probabilities by single life probabilities:

$$p_{xyz\cdots} = p_{x+t} \Rightarrow a_{xyz\cdots} = a_{x+t}$$

Based on what you are given, you have

$$\ddot{a}_{x:x+5} = \ddot{a}_{x+5+3} = \ddot{a}_{x+8}$$

Now you should write down the present value of the normal form and optional form benefit. Their present values are equal, since the benefits are actuarially equivalent:

1-1-2000 Partic age = 65 Spouse age = 60

PV normal form = $12(1500)\ddot{a}_{65}^{(12)}$

PV optional form $= 12P\left[\ddot{a}_{\overline{10}|}^{(12)} + {}_{10|}\ddot{a}_{65}^{(12)} + .75\left({}_{10|}\ddot{a}_{60}^{(12)} - {}_{10|}\ddot{a}_{60:65}^{(12)}\right)\right]$

$= 12P\left[\ddot{a}_{\overline{10}|}^{(12)} + {}_{10|}\ddot{a}_{65}^{(12)} + .75\left({}_{10|}\ddot{a}_{60}^{(12)} - {}_{10|}\ddot{a}_{68}^{(12)}\right)\right]$ Using Gompertz

$$12(1500)\frac{N_{65}^{(12)}}{D_{65}} = 12P\left[\ddot{a}_{\overline{10}|}^{(12)} + \frac{N_{75}^{(12)}}{D_{65}} + .75\left(\frac{N_{70}^{(12)}}{D_{60}} - \frac{N_{78}^{(12)}}{D_{68}}\right)\right]$$

$$12(1500)\frac{842{,}966}{96{,}496} = 12P\left[7.28714 + \frac{222{,}026}{96{,}496} + .75\left(\frac{455{,}486}{147{,}589}\right) - .75\left(\frac{134{,}863}{73{,}034}\right)\right]$$

$$1500(8.7358) = P\left[7.28714 + 2.3009 + .75(3.0862 - 1.8466)\right]$$

$$P = \frac{8.7358\,(1500)}{10.5177}$$

$$= 1{,}246$$

D

Spring 2000 EA-1A

24 The key to working this problem is understanding how to write down the amount of interest and principal in any payment to a loan:

Pmt #	Payment	Interest	Principal	O/S Loan
0				$P\,a_{\overline{100}|i} = P(1-v^{100})/i$
1	P	$P(1-v^{100})$	Pv^{100}	$P\,a_{\overline{99}|i}$
2	P	$P(1-v^{99})$	Pv^{99}	$P\,a_{\overline{98}|i}$

You need to solve for the payment number where the principal exceeds the interest. If n is the payment number then

$$Pv^{101-n} \geq P(1-v^{101-n})$$
$$2Pv^{101-n} \geq P$$
$$v^{101-n} \geq .5$$
$$(101-n)\log v \geq \log .5$$
$$101-n \leq (\log .5)/(\log v)$$

The inequality reversed because $(\log v)$ is a negative number! You are told the annual interest rate is 8%, compounded continuously, which means $1+i = e^{\delta} = e^{.08} = 1.0833$. Since you have quarterly payments, the rate per quarter is $1+j = (1.0833)^{1/4} = 1.0202$. Now solve for n

$$101-n \leq (\log .5)/(\log .9802)$$
$$\leq 34.66$$

$66.34 \leq n$ $\quad\therefore$ n must be 67th payment

D

Spring 2000 EA-1A

25 The key to working this problem is primarily understanding the symbols and notation:

$a_{x|y}$ = reversionary annuity to y after death of x
$= a_y - a_{xy}$

$a_{\overline{zw}}$ = joint and last survivor annuity, payable as long as at least one of z and w survive
$= a_z + a_w - a_{zw}$

$$a_{x|\overline{zw}} = a_{\overline{zw}} - a_{x:\overline{zw}}$$

$$a_{65|\overline{50:20}} = a_{\overline{50:20}} - a_{65:\overline{50:20}}$$
$$= (a_{50} + a_{20} - a_{50:20}) - (a_{65:50} + a_{65:20} - a_{65:50:20})$$
$$= (12.56 + 14.79 - 12.51) - (9.48 + 9.68 - 9.44)$$
$$= 14.84 - 9.72$$
$$= 5.12$$

B