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SPRING 2002 EA-1
EXAM SOLUTIONS

Revision History:

04/12/06	Added alternate solutions for problem 8 and problem 18	
02/15/06	Corrected problem 08	Some exponents on page 2 of solution were incorrect (30 S/B 29)
05/03/04	Corrected problem 05	Evaluation point on page 2 was incorrect, should be at time 4
05/03/04	Modified problem 08	Added alternate solution
05/03/04	Corrected problem 14	Force of mortality on page 1 incorrectly showed age 42.4

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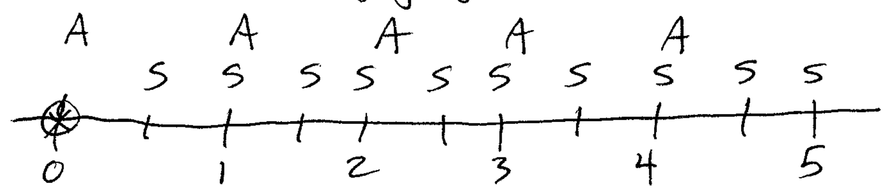
2002 EA-1

1 This is a typical loan problem. You should write down the series of payments on a time line diagram:



If i is the annual rate of interest, then $A(\ddot{a}_{\overline{5}|i}) = 75{,}000$.
If j is the semi-annual rate of interest, then $S(a_{\overline{10}|j}) = 75{,}000$.
I always convert the interest rate so that it matches the payment period. The next step is calculation of i and j

$$d^{(4)} = \frac{76.225}{1000} \qquad 1+i = \left(1-\frac{d^{(m)}}{m}\right)^{-m} = (1-.0191)^{-4} = 1.0800$$

$$1+i = (1+j)^2 \qquad 1+j = (1.08)^{.5} = 1.0392$$

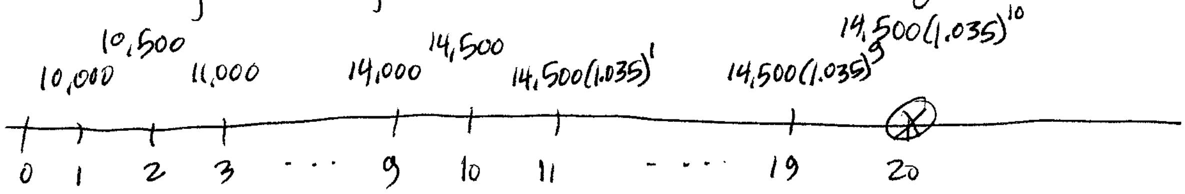
When calculating annuities with the HP-12C calculator, I always calculate annuities immediate. To calculate an annuity due, I multiply by $(1+i)$. I find this reduces the chance for error caused when changing the calculator from BEG to END when calculating annuities.

$$A = 75{,}000 / \ddot{a}_{\overline{5}|.0800} = \frac{75{,}000}{1.08(3.9927)} = 17{,}393$$

$$S = 75{,}000 / a_{\overline{10}|.0392} = \frac{75{,}000}{8.1421} = 9{,}211$$

$B = 2S = 18{,}423$ $\qquad |A-B| = 1{,}030$ $\qquad$ C

2 The key to this problem is first, carefully writing the correct series of payments on a time line diagram. Then, you should either know the formula for an increasing annuity, or be able to derive the formula.



It will be easier to deal with the first 10 years of payments separately from the second ten years. Let A represent the accumulated value at time 20 for the first 10 payments, and let B be the value for the second 10 payments:

$$A = \left[10{,}000(s_{\overline{10}|.07}) + 500(Is)_{\overline{9}|.07}\right](1.07)^{10}$$

$$B = 14{,}500(1.035)^1(1.07)^9 + \dots + 14{,}500(1.035)^9(1.07) + 14{,}500(1.035)^{10}$$

To evaluate B, there is a standard technique used for geometrically increasing payments. You should factor out the last term:

$$B = 14{,}500(1.035)^{10}\left[\left(\frac{1.07}{1.035}\right)^9 + \dots + \frac{1.07}{1.036} + 1\right]$$

$$= 14{,}500(1.035)^{10}\, s_{\overline{10}|j} \qquad \text{where } 1+j = \frac{1.070}{1.035} = 1.0338$$

$$= 14{,}500(1.4106)(10.3185)$$

$$= 238{,}642$$

(next page)

(2) Now you need to evaluate A, the value of the 1st 10 payments:

$$A = \left[10{,}000\, s_{\overline{10}|.07} + 500(Is)_{\overline{9}|.07}\right](1.07)^{10}$$

You can either derive the value of $(Is)_{\overline{n}|}$, or use the formula

$$(Is)_{\overline{n}|i} = (1+i)^n (Ia)_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|i} - n}{i} = \frac{(1+i)s_{\overline{n}|i} - n}{i}$$

$$A = \left[10{,}000\,(s_{\overline{10}|.07}) + 500\left(\frac{(1.07)\,s_{\overline{9}|.07} - 9}{.07}\right)\right](1.07)^{10}$$

$$= \left[10{,}000(13.8164) + 500\left[\frac{(1.07)(11.9780) - 9}{.07}\right]\right](1.9672)$$

$$= (138{,}164 + 27{,}260)\,1.9672$$

$$= 325{,}416$$

Finally, $A + B = 238{,}642 + 325{,}416$

$$= 564{,}058$$

C

3 This problem tests your knowledge of the formulas for annuity values that are paid more frequently than annually:

$$\ddot{s}_{\overline{n}|i}^{(m)} = \frac{(1+i)^n - 1}{d^{(m)}} = \frac{d}{d^{(m)}}\ddot{s}_{\overline{n}|i}$$

$$\ddot{a}_{\overline{n}|i}^{(m)} = \frac{1-v^n}{d^{(m)}} = \frac{d}{d^{(m)}}\ddot{a}_{\overline{n}|i}$$

You need to write the formulas for $\ddot{s}_{\overline{2n}|}^{(m)}$ and $\ddot{s}_{\overline{4n}|}^{(m)}$, then see what is required to evaluate the second one:

$$\ddot{s}_{\overline{2n}|}^{(m)} = \frac{(1+i)^{2n} - 1}{d^{(m)}} \qquad \ddot{s}_{\overline{4n}|}^{(m)} = \frac{(1+i)^{4n} - 1}{d^{(m)}}$$

$$180.24943 = \frac{(1+i)^{2n} - 1}{.08}$$ based on the information given

You can solve for the value of $(1+i)^{2n}$, then use that to calculate the value of $\ddot{s}_{\overline{4n}|}^{(m)}$

$$(1+i)^{2n} = .08(180.24943) + 1$$
$$= 15.42$$
$$(1+i)^{4n} = [(1+i)^{2n}][(1+i)^{2n}]$$
$$= (15.42)^2$$

$$\ddot{s}_{\overline{4n}|}^{(m)} = \frac{(15.42)^2 - 1}{.08} = 2959.69$$ B

4 The first step in the problem is calculation of the monthly interest rate. I always try to match the interest compounding period to the payment frequency.

You are given the nominal discount rate as 8%, compounded quarterly. Here is the relationship between the monthly interest rate j and the annual interest i

$$1+i = \left(1-\frac{d^{(4)}}{4}\right)^{-4} = (1+j)^{12} \quad \Rightarrow (1+j) = (1.0842)^{1/12}$$
$$= 1.006757$$
$$= (1-.02)^{-4}$$
$$= 1.0842 \qquad\qquad j = .6757\% \text{ per month}$$

The 20 year annuity has 240 monthly payments:

After payment 0, the present value is $M(a_{\overline{240}|}j)$
" " 1, " " " " $M(a_{\overline{239}|}j)$
" " 43, " " " " $M(a_{\overline{197}|}j) = X$
" " N, " " " " $M(a_{\overline{240-N}|}j) = \dfrac{X}{2}$

$$a_{\overline{240-N}|}j = \frac{1}{2}(a_{\overline{197}|}j) = \frac{1}{2}(108.72) = 54.36$$

You can use your calculator to solve for the value of N. If you have the HP-12C, it always rounds the number of payments <u>up</u>. You can calculate the exact value as follows $a_{\overline{240-N}|}.6757\% = \dfrac{1-v^{(240-N)}}{.006757} = 54.36$

This produces a value of 172.02 for N.

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(4) The HP-12C gives a value of 68 for 240−N, which corresponds to N=172. You should check the present values for 67 and 68 to be sure that you have the right answer:

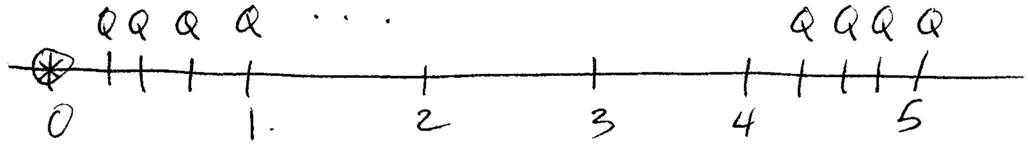
$$N=172 \qquad a_{\overline{240-N}|.6757\%} = a_{\overline{68}|.6757\%} = 54.37 > \frac{X}{2}$$

$$N=173 \qquad a_{\overline{240-N}|.6757\%} = a_{\overline{67}|.6757\%} = 53.74 < \frac{X}{2}$$

The correct answer is E

5 The key to working this problem is understanding how the sinking fund works to repay Loan #2. It is described correctly in the problem. You must make monthly payments of interest on the original 10,000 loan. The sinking fund will accumulate at 9% interest to pay off the 10,000 loan amount at the end of 4 years.

For loan 1, you need to calculate the quarterly interest rate equivalent to 8% per annum. Then you can calculate the monthly payment for Loan 1.



$A = 20Q$

$Q\, a_{\overline{20}|j} = 10{,}000 \qquad (1+j)^4 = 1.08 \;\Rightarrow\; \text{quarterly } j = 1.943\%$

$$Q = \frac{10{,}000}{a_{\overline{20}|.01943}}$$

$= 608.19 \qquad \therefore A = 20(608.19) = 12{,}163.76$

For loan 2, the monthly interest rate equivalent to 8% per annum is $.6434\% = (1.08)^{\frac{1}{12}} - 1$. The monthly interest payment on the loan is

$10{,}000\,(.6434\%) = 64.34$

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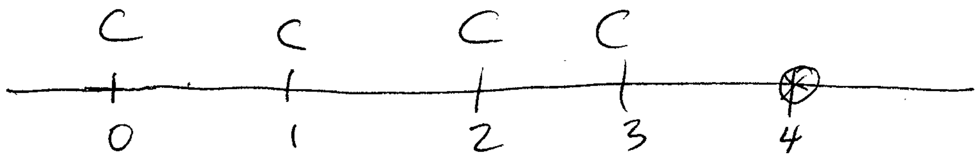
Revised 05/03/04

(5) under loan 2, the sinking fund payments are made at the beginning of each year for four years:



$$C(\ddot{s}_{\overline{4}|.09}) = 10{,}000$$

$$C = \frac{10{,}000}{1.09(s_{\overline{4}|.09})}$$

$$= 2{,}006.13$$

I use immediate annuity calculations with the HP-12C calculator

B = Sum of monthly interest payments, plus annual sinking fund payments for loan 2

$$= 48(64.34) + 4(2{,}006.13)$$

$$= 11{,}112.87$$

$$A - B = 12{,}163.76 - 11{,}112.87$$

$$= 1{,}050.89$$

$$|A - B| = 1{,}050.89$$

D

6 The key to working this problem is being able to write down the amortization schedule for the loan payments. Based on annual payments of P, the loan of 10,000 equals $P(a_{\overline{40}|.07})$. Here is the loan amortization schedule:

Year	Payment	Interest	Principal	O/S Loan
0				$Pa_{\overline{40}\|.07} = P\frac{(1-v^{40})}{i}$
1	P	$P(1-v^{40})$	Pv^{40}	$Pa_{\overline{39}\|.07}$
2	P	$P(1-v^{39})$	Pv^{39}	$Pa_{\overline{38}\|.07}$
⋮	⋮	⋮	⋮	⋮
39	P	$P(1-v^{2})$	Pv^{2}	$Pa_{\overline{1}\|.07}$
40	P	$P(1-v^{1})$	Pv^{1}	0

A: Interest in the even payments:
$$P(1-v^{39}) + P(1-v^{37}) + \ldots + P(1-v^{1})$$

B: Principal in the odd payments:
$$Pv^{40} + Pv^{38} + \ldots + Pv^{2}$$

One way to work the problem is to separately determine expressions for the sums. Another way is to recognize that the sums are similar. You can express the value of A in terms of B:

$$1.07A = 1.07\left[P(v-v^{40}) + P(v-v^{38}) + \ldots + P(v-v^{2})\right]$$
$$= 1.07\left[20Pv - B\right]$$

(6) The fastest way to evaluate A and B is to replace the 7% per annum interest rate with a biennial interest rate of $14.49\% = (1.07)^2 - 1$:

$$B = P(1.07)^{-40} + P(1.07)^{-38} + \dots + P(1.07)^{-2}$$
$$= P(1.1449)^{-20} + P(1.1449)^{-19} + \dots + P(1.1449)^{-1}$$
$$= P\, a_{\overline{20}|14.49\%}$$

$$P = \frac{10{,}000}{a_{\overline{40}|.07}} = \frac{10{,}000}{13.3317} = 750.09$$

$$B = 750.09(6.4404)$$
$$= 4{,}830.92$$

$$A = 1.07\left[\frac{20(750.09)}{1.07} - 4830.92\right]$$
$$= 1.07(14{,}020.40 - 4830.92]$$
$$= 9{,}832.74$$

$$A + B = 14{,}663.66$$

D

7 This is a typical exam problem on both the concept of refinancing a mortgage, as well as the amortization schedule. The concepts are quite similar for loan problems.

The initial mortgage is for 80% of 120,000=96,000. The monthly interest rate is .08/12=.6667%.

The monthly payment is $96{,}000/a_{\overline{360}|.6667\%}=704.41$. You can write down a few lines of the amortization schedule to calculate the interest in the 100th payment:

Time	Payment	Principal	Interest	O/S Mortgage
0	—	—	—	$704.41(a_{\overline{360}|.6667\%})$
1	704.41	$704.41v^{360}$	$704.41(1-v^{360})$	$704.41(a_{\overline{359}|.6667\%})$
⋮	⋮	⋮	⋮	⋮
100	704.41	$704.41v^{261}$	$704.41(1-v^{261})$	$704.41(a_{\overline{260}|.6667\%})$

The interest in the 100th payment is $704.41(1-(1.006667)^{-261})$, which equals $580.06=A$.

The amount of the second mortgage equals the O/S payments after the 180th payment:

$704.41(a_{\overline{180}|.6667\%})=73{,}710$

(next page)

(7) The monthly interest rate for the second mortgage is $.075/12 = .6250\%$. The monthly payment is $73{,}710 / a_{\overline{120}|.6250\%} = 874.95$.

You can determine the principal in the 100th payment by looking the the new amortization schedule

Time	Payment	Principal	Interest	O/S mortgage
0	—	—	—	$874.95(a_{\overline{120}|.6250\%})$
1	874.95	$874.95v^{120}$	$874.95(1-v^{120})$	$874.95(a_{\overline{119}|.6250\%})$
⋮	⋮	⋮	⋮	⋮
100	874.95	$874.95v^{21}$	$874.95(1-v^{21})$	$874.95(a_{\overline{20}|.6250\%})$

The principal in the 100th payment is $874.95(1.006250)^{-21}$, which equals $767.65 = B$.

$A + B = 1{,}347.71$ C

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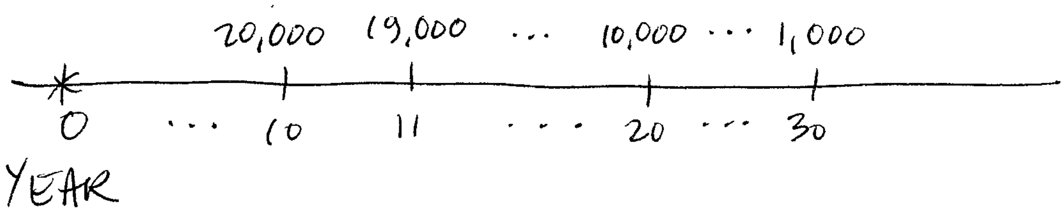
8 This is a messy serial bond problem. The bond coupons are 3% every six months, but the face amount of each bond is different. It does not matter whether you use Makeham's formula, since none of the results are easy to evaluate.

To write the present value of the bonds, you should convert the yield to a semi-annual basis, which matches the frequency of the coupons:

$(1.07)^{\frac{1}{2}} - 1 \Rightarrow$ semi annual rate $= 3.441\%$

Write down the face values of the bonds on a time line diagram first. Then you can write down the value of each bond at the 3.441% semi-annual yield rate:



PV of 1st bond: $Fr(a_{\overline{n}|i}) + K$

$$= .03(20{,}000)\,a_{\overline{20}|3.441\%} + 20{,}000\,(1.03441)^{-20}$$

PV of 2nd bond $= .03(19{,}000)\,a_{\overline{22}|3.441\%} + 19{,}000\,(1.03441)^{-22}$

PV of 20th bond $= .03(1{,}000)\,a_{\overline{60}|3.441\%} + 1{,}000\,(1.03441)^{-60}$

(next page)

Solution Revised 2/15/06

(8) The total series of bonds is difficult to evaluate, based on the fact that the coupon payments changes by 2 for each bond. It appears that the easiest way to evaluate the series is to write the present values based on a 7% annual rate:

$$PV \text{ of } 1^{st} \text{ bond} = .03(20{,}000)\left[\frac{1-(1.03441)^{-20}}{.03441}\right] + 20{,}000(1.03441)^{-20}$$

$$= \frac{.03}{.03441}(20{,}000)\left[1-(1.07)^{-10}\right] + 20{,}000(1.07)^{-10}$$

$$PV \text{ of } 2^{nd} \text{ bond} = \frac{.03}{.03441}(19{,}000)\left[1-(1.07)^{-11}\right] + 19{,}000(1.07)^{-11}$$

$$PV \text{ of } 20^{th} \text{ bond} = \frac{.03}{.03441}(1{,}000)\left[1-(1.07)^{-30}\right] + 1{,}000(1.07)^{-29}$$

$$PV \text{ of entire series} = .87189\begin{bmatrix} 20{,}000 & -20{,}000(1.07)^{-10} \\ +19{,}000 & -19{,}000(1.07)^{-11} \\ + \cdots & - \cdots \\ +1{,}000 & -1{,}000(1.07)^{-29} \end{bmatrix} \begin{matrix} +20{,}000(1.07)^{-10} \\ +19{,}000(1.07)^{-11} \\ + \cdots \\ +1{,}000(1.07)^{-29} \end{matrix}$$

$$= .87189\left[20{,}000(21/2) - 1{,}000(1.07)^{-9}(Da_{\overline{20}|.07})\right] + 1{,}000(1.07)^{-9}(Da_{\overline{20}|.07})$$

$$Da_{\overline{n}|i} = (n+1)a_{\overline{n}|i} - Ia_{\overline{n}|i}$$

$$Da_{\overline{20}|.07} = 21\,a_{\overline{20}|.07} - (\ddot{a}_{\overline{20}|.07} - 20v^{20})/.07$$

$$= 134.3712$$

$$PV \text{ of entire series} = .87189(210{,}000) + 1{,}000(1.07)^{-9}(134.3712)(1-.87189)$$

$$= 183{,}097 + 9{,}363$$

$$= 192{,}460 \quad \text{E}$$

(revised 04/12/06)
Solution Added 05/03/04

(8) Continued

Since this is a serial bond problem, it makes more sense to value the bonds using Makeham's formula:

$$P = K + \frac{g}{i}(C-K)$$

$$= Cv^n + \left(\frac{Fr}{Ci}\right)(C - Cv^n)$$

C: redemption value
F: face value
r: coupon rate
i: yield rate

PV of bond #1

$$P = 20{,}000(1.03441)^{-20} + \frac{20{,}000(.03)}{20{,}000(.03441)}(20{,}000 - 20{,}000(1.03441)^{-20})$$

$$= 20{,}000\left[(1.07)^{-10} + \frac{.03}{.03441}(1-(1.07)^{-10})\right]$$

I replaced the semi-annual yield rates of 3.441% with annual yield rates of 7%. This is a much easier result to work with. Now you can write similar formulas for the rest of the bonds:

Bond #2

$$P = 19{,}000\left[(1.07)^{-11} + .87189(1-(1.07)^{-11})\right]$$

⋮

Bond #20

$$P = 1{,}000\left[((1.07)^{-29} + .87189(1-(1.07)^{-29})\right]$$

(next page)

(8) continued

The price for the series of bonds can be written as:

$$20{,}000\left[(1.07)^{-10} + .87189\left(1-(1.07)^{-10}\right)\right]$$
$$+ 19{,}000\left[(1.07)^{-11} + .87189\left(1-(1.07)^{-11}\right)\right]$$
$$+ \cdots \qquad + \cdots$$
$$+ 1{,}000\left[(1.07)^{-29} + .87189\left(1-(1.07)^{-29}\right)\right]$$

$$= 20{,}000(.87189) + .12811(20{,}000)(1.07)^{-10}$$
$$+ 19{,}000(.87189) + .12811(19{,}000)(1.07)^{-11}$$
$$+ \cdots \qquad + \cdots$$
$$+ 1{,}000(.87189) + .12811(1{,}000)(1.07)^{-29}$$
$$= \quad P \quad + \quad Q$$

$$P = .87189\,\frac{(20)(21)}{2}\,(1{,}000)$$
$$= 183{,}097$$

$$Q = (1.07)^{-9}(.12811)\left[20{,}000v + 19{,}000v^2 + \ldots + 1{,}000v^{20}\right]$$
$$= (1.07)^{-9}(.12811)\,1{,}000\left[Da_{\overline{20}|.07}\right]$$
$$= (1.07)^{-9}(.12811)\,1{,}000\left(\frac{20 - a_{\overline{20}|.07}}{.07}\right)$$
$$= .5439(.12811)(1{,}000)(134.37)$$
$$= 9{,}363$$

Total price of bonds

$P + Q = 192{,}460$ E

9 The key to working this problem is knowing that the book value of the bond is the same as the amortized value. The amortized value of a bond at any time is calculated using the standard price formula for a bond, based on the remaining coupon period.

Since the coupon rate is convertible semi-annually, the bond has semi-annual coupons of $.09/2 = 4.5\%$ of the face value. The annual yield rate should be converted to a semi-annual basis, so that it matches the coupon frequency:

semi-annual $j \Rightarrow (1+j)^2 = 1.1025 \quad j = 5.0\%$

At time 0, the purchase price of the bond can be calculated using this formula:

$P_0 = .045(1{,}000)\, a_{\overline{20}|.05} + 1{,}050(1.05)^{-20}$

At the end of the first year, you have 2 less coupons:

$P_1 = .045(1{,}000)\, a_{\overline{18}|.05} + 1{,}050(1.05)^{-18}$

The change in value during the third year is $P_3 - P_2$:

$P_2 = 45\, a_{\overline{16}|.05} + 1{,}050(1.05)^{-16} = 968.72$

$P_3 = 45\, a_{\overline{14}|.05} + 1{,}050(1.05)^{-14} = 975.76$

$P_3 - P_2 = 7.04$ B

10 The key to working this problem is interpretation of how the lender would "repay the loan in full". You should calculate the present value of the remaining loan payments at the loan interest rate, and assume that amount is paid.

The monthly loan interest rate is 7%/12 = .5833%, since you are told that the 7% is compounded monthly. Based on the loan amount of 100,000, you can calculate the monthly loan payment

$$\text{Payment} = \frac{100{,}000}{a_{\overline{240}|.5833\%}} = 775.30$$

At 1-1-02, two years of payments have been made. There are 216 payments remaining (240-24). You should compare the present value of the payments at the 8% yield rate to the 90,000 purchase price. The easiest way to work the problem is to test the dates given.

Since the yield rate is given as 8% compounded monthly, you should use a monthly rate of .6667%, which is .08/12. For the first answer to test, I'll start in the middle with answer C. Depending on the result, I'll either choose B or D next.

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(10) Use the 1-1-2011 repayment date from answer C. There will be 9 years (2011-2002) of monthly loan payments, plus the lump sum loan repayment at 1-1-2011: 9 years ⇒ 108 payments made, 216 − 108 = 108 remaining

$$PV = 775.30(a_{\overline{108}|.6667\%}) + (1.006667)^{-108}[775.30(a_{\overline{108}|.5833\%})]$$
$$= 59{,}553 + 30{,}247 = 775.30(76.8125) + .4879(775.30)(79.9598)$$
$$= 89{,}799.96$$

Since the present value is less than 90,000, the yield based on this repayment date is less than 8% on a monthly basis. The next answer to try should be 1-1-2010. The annuity at the .5833% will have 12 additional payments, and the annuity at .6667% will have 12 fewer payments. The net result should increase the PV.

1-1-2010: 8 years ⇒ 96 payments made, 216 − 96 = 120 remaining

$$PV = 775.30(a_{\overline{96}|.6667\%}) + (1.006667)^{-96}[775.30(a_{\overline{120}|.5833\%})]$$
$$= 775.30(70.7380) + .5284(775.30)(86.1264)$$
$$= 54{,}843 + 35{,}284$$
$$= 90{,}127$$

Since the present value is greater than 90,000, the yield based on this repayment date exceeds 8% (convertible monthly).

B

11 This is a fairly typical exam question on interest and discount rates. The key to working this problem is carefully handling the information regarding the rate of interest (or discount) and how frequently the rate is compounded.

To value S_1, you are given the nominal interest rate as 8% per annum, compounded quarterly. Since you are valuing a series of monthly payments, you should determine the equivalent monthly interest rate:

$$1+i = \left(1+\frac{i^{(4)}}{4}\right)^4 = \left(1+\frac{.08}{4}\right)^4 = 1.0824$$

monthly rate $j \Rightarrow (1+j)^{12} = 1.0824, \quad j = .6623\%$

$$S_1 = 500\left(s_{\overline{12}|.6623\%}\right) = 500(12.4469)$$
$$= 6{,}223.45$$

To value A_1, you are given the nominal discount rate as 6% per annum, compounded semi-annually. I find it easier to simply convert this to annual interest rates, which is less confusing to me:

$$1+i = \left(1-\frac{d^{(2)}}{2}\right)^{-2} = \left(1-\frac{.06}{2}\right)^{-2} = 1.0628$$

$$A_1 = S_1 / 1.0628$$
$$= 5855.64$$

(next page)

2002

(11) To value S_2, you are given the nominal discount rate as 6% per year, convertible monthly. Since you are given a series of quarterly payments, you should determine the equivalent quarterly interest rate:

$$1+i = \left(1-\frac{d^{(12)}}{12}\right)^{-12} = \left(1-\frac{.06}{12}\right)^{-12} = 1.0620$$

quarterly rate $k \Rightarrow (1+k)^4 = 1.0620$, $k = 1.515\%$

$$S_2 = 1{,}500\left(s_{\overline{4}|1.515\%}\right) = 1500(4.0918)$$
$$= 6{,}137.74$$

To value A_2, you have a nominal annual rate of $P\%$, convertible once every two years. If you think about the relationship for a semi-annual interest rate, you should be able to write one for the biennial rate P:

$$1+i = \left(1+\frac{semi}{2}\right)^2 = (1+2P)^{.5}$$

Now you can equate A_2 and A_1, and solve for P:

$$A_2 = \frac{6137.74}{1+i} = \frac{6137.74}{(1+2P)^{.5}} = A_1 = 5855.64$$

$$(1+2P)^{.5} = 6137.74/5855.64 = 1.0482$$
$$1+2P = 1.0987$$
$$P = 4.934\%$$

(E) (inside the implied range)

12 This is the first problem on measurement of mortality asked on the EA-1 exam. If you try to set up the usual definitions for q_x based on the observed lives, you get the wrong answer range.

The key to the problem is knowing that the "moment estimate" is defined in the Survival Models textbook as the value of q_x taking into account the actual exposure of lives age x until death during the observation period.

Based on Bowers Table 3.6.1, various results can be derived for the Balducci mortality assumption:

$${}_{1-t}q_{x+t} = (1-t)q_x$$

and

$${}_tq_x = \frac{t(q_x)}{1-(1-t)q_x}$$

Of the original 100 lives, 40 are exposed for 3/4 of a year. The remaining 60 lives are exposed for the full year. You can set up the formula for the 23 deaths observed:

$$60\,q_x + 40\left({}_{\frac{3}{4}}q_x\right) = 23$$

$$60q_x + 40\left(\frac{.75\,q_x}{1-(1-.75)q_x}\right) = 23$$

(next page)

(12) $$60q_x + \frac{30q_x}{1-.25q_x} = 23$$

$$60q_x(1-.25q_x) + 30q_x = 23(1-.25q_x)$$

$$60q_x - 15(q_x)^2 + 30q_x = 23 - 5.75q_x$$

$$0 = 15(q_x)^2 - 95.75q_x + 23$$
$$= (q_x)^2 - 6.3833q_x + 1.5333$$

Now use the quadratic equation to solve for q_x.
Based on $ax^2+bx+c=0$, you have

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$q_x = \frac{6.3833 \pm \sqrt{(6.3833)^2 - 4(1)(1.5333)}}{2}$$

$$= \frac{6.3833 \pm 5.8833}{2}$$

$$= 6.13 \text{ or } .250$$

The value of 6.13 is a spurious result for q_x, so the correct value is .250.

B

13 The key to working this problem is knowing the formulas that relate l_x, ${}_tp_x$, and μ_x:

$$\mu_x = \left(-\frac{d}{dx} l_x\right)\frac{1}{l_x}$$

$$ {}_np_x = e^{-\int_0^n \mu_{x+t}\,dt} = \frac{l_{x+n}}{l_x}$$

First, you can use the given formula for l_x to calculate the value of μ_{41} in mortality table A:

$$\mu_x = \left[\frac{-d}{dx}(20{,}000 - 100x - x^2)\right]\frac{1}{20{,}000-100x-x^2}$$

$$= \frac{-(-100-2x)}{20{,}000-100x-x^2}$$

$$\mu_{41} = \frac{100+2(41)}{20{,}000-100(41)-(41)^2} = \frac{182}{14{,}219} = .0128$$

Now you can use this value of .0128 as the constant force in mortality table B. Since you are given the value of l_{45}, use the formula for ${}_4p_{41}$ to solve for l_{41}:

$${}_4p_{41} = l_{45}/l_{41} = e^{-\int_0^4 .0128\,dt}$$

$$= e^{-4(.0128)}$$

$$= .950089$$

$$l_{41} = l_{45}/.9501 = 105{,}253$$

C

Revised 05/03/04

14 The key to working this problem is knowing several formulas based on the assumption of uniform distribution of decrements over a single year of age. The following formulas are shown in Bowers table 3.6.1

U.D.D. $_tq_x = t(q_x)$ for $0 \le t \le 1$

$$\mu_{x+t} = \frac{q_x}{1-t(q_x)} \qquad {}_{1-t}q_{x+t} = \frac{(1-t)q_x}{1-t(q_x)} \qquad {}_yq_{x+t} = \frac{(y)q_x}{1-t(q_x)}$$

The relationship between l_{40} and l_{43} is based on survival probabilities for three years: $l_{43} = l_{40}(p_{40})(p_{41})(p_{42})$. You must derive the values of p_{40}, p_{41} and p_{42} based on the information given in the problem

$${}_{.5}q_{40.4} = .025 = \frac{.5(q_{40})}{1-.4(q_{40})} \Rightarrow .025 - .01(q_{40}) = .5(q_{40})$$

$$\therefore p_{40} = .9510$$

$${}_{.9}p_{41} = .955 = 1-.9(q_{41}) \Rightarrow .9q_{41} = .045$$

$$\therefore p_{41} = .95$$

$$\mu_{42.2} = .05 = \frac{q_{42}}{1-.2(q_{42})} \Rightarrow .05 - .01(q_{42}) = q_{42}$$

$$\therefore p_{42} = .9505$$

$$l_{40} = \frac{l_{43}}{p_{40}(p_{41})p_{42}} = \frac{100{,}000}{.9510(.95)(.9505)}$$

$$= 116{,}454$$

B

15 The key to working this problem is writing down the expression for the present value correctly, and seeing how to evaluate it:

$$20{,}000\,\ddot{a}_{65} = X\ddot{a}_{65} + \underbrace{v^2 p_{65}\, q_{66}}_{\text{Survive 2002, die in 2003}} \underbrace{[X + (.95v)^1 X + (.95v)^2 X + \dots + (.95v)^9 X]}_{10\text{ payments}}$$

To value the payments in the brackets, you will cancel out the rate of interest and the decrease factor:

$$X[1 + .95v + (.95v)^2 + \dots + (.95v)^9]$$
$$= X\left[1 + \frac{.95}{1.07} + \dots + \left(\frac{.95}{1.07}\right)^9\right]$$
$$= X\,\ddot{a}_{\overline{10}|j} \quad \text{where } 1+j = \frac{1.07}{.95} = 1.1263$$
$$= X\,\ddot{a}_{\overline{10}|12.63\%}$$
$$= X(6.2027)$$

Now you can go back and solve for X from the 1st equation

$$20{,}000\,\ddot{a}_{65} = X\ddot{a}_{65} + v^2 p_{65}\, q_{66}[X(6.2027)]$$
$$= X[\ddot{a}_{65} + v^2 p_{65}\, q_{66}(6.2027)]$$
$$X = \frac{20{,}000\,\ddot{a}_{65}}{\ddot{a}_{65} + v^2 p_{65}\, q_{66}(6.2027)}$$
$$= \frac{20{,}000(10.3316)}{10.3316 + (1.07)^{-2}(.9887)(1-.9873)(6.2027)}$$
$$= 19{,}869.17$$

E

16 The key to working this problem is understanding the definition of m_x, as well as ${}_n m_x$:

$$m_x = \frac{d_x}{L_x} \qquad {}_n m_x = \frac{{}_n d_x}{{}_n L_x} = \frac{d_x + d_{x+1} + \ldots + d_{x+n-1}}{L_x + L_{x+1} + \ldots + L_{x+n-1}}$$

m_x is the central death rate, which measures the death rate over the year of age x. ${}_n m_x$ measures the death rate over a period of n years.

Based on the information given, you need to solve for the value of l_{37}. That would allow you to calculate the value of $l_{36.5}$:

$$\frac{l_{36.5}}{l_{36}} = {}_{.5}p_{36}$$

Under constant force of mortality assumption:

$$\mu_{x+t} = \mu \quad \text{for } 0 \le t \le 1$$

$$ {}_t p_x = e^{-t\mu} = (p_x)^t$$

$$\therefore {}_{.5}p_{36} = (p_{36})^{1/2} = \left(\frac{l_{37}}{l_{36}}\right)^{\frac{1}{2}}$$

(next page)

(16)

$$2m_{35} = \frac{d_{35}+d_{36}}{L_{35}+L_{36}}$$

$$.0404 = \frac{300+d_{36}}{9851+9456}$$

$$d_{36} = .0404(9851+9456)-300$$
$$=480.00$$

$$l_{37} = l_{36}-d_{36}$$
$$=9700-480$$
$$=9220$$

$$.5p_{36} = \left(\frac{9220}{9700}\right)^{.5}$$
$$=.9749$$

$$l_{36.5} = l_{36}(.5p_{36})$$
$$=9700(.9749)$$
$$=9456.96$$

D

17 The key to working this problem is figuring out the relationships between the expectation of life factors. Eventually, you have to figure out a way to solve the various equations to calculate e_{70}:

$$\begin{aligned} e_{70} &= {}_1p_{70} + {}_2p_{70} + {}_3p_{70} + \dots \\ &= e_{70:\overline{5|}} + {}_5p_{70}(e_{75}) \\ &= e_{70:\overline{5|}} + {}_5p_{70}(e_{75:\overline{5|}}) + {}_{10}p_{70}(e_{80}) \end{aligned}$$

At this point, you know the values for the expectations but not the values of the probability terms. You should write down similar equations that use more of the expectations (for which you know the value).

$$e_{70:\overline{15|}} = e_{70:\overline{5|}} + {}_5p_{70}(e_{75:\overline{5|}}) + {}_{10}p_{70}(e_{80:\overline{5|}})$$

$$e_{75:\overline{10|}} = e_{75:\overline{5|}} + {}_5p_{75}(e_{80:\overline{5|}})$$

At this point, you are done! The key is that you can write ${}_{10}p_{70}$ as ${}_5p_{70}({}_5p_{75})$. You can use the last equation to solve for ${}_5p_{75}$, since you know the value of the expectations. Then you can use the prior equation to solve for the value of ${}_5p_{70}$. Finally, you can use the probability values to calculate e_{70} in the original equation.

(next page)

(17) $e_{75:\overline{10|}} = e_{75:\overline{5|}} + {}_5p_{75}(e_{80:\overline{5|}})$

$7.70883 = 4.43230 + {}_5p_{75}(4.08531)$

$${}_5p_{75} = \frac{7.70883 - 4.43230}{4.08531}$$

$= .8020$

$e_{70:\overline{15|}} = e_{70:\overline{5|}} + {}_5p_{70}(e_{75:\overline{5|}}) + {}_5p_{70}({}_5p_{75})(e_{80:\overline{5|}})$

$e_{70:\overline{15|}} - e_{70:\overline{5|}} = {}_5p_{70}\left[e_{75:\overline{5|}} + {}_5p_{75}(e_{80:\overline{5|}})\right]$

$${}_5p_{70} = \frac{e_{70:\overline{15|}} - e_{70:\overline{5|}}}{e_{75:\overline{5|}} + {}_5p_{75}(e_{80:\overline{5|}})}$$

$$= \frac{11.45220 - 4.66234}{4.43230 + .8020(4.08531)}$$

$= 6.7899/7.7088$

$= .8808$

$e_{70} = e_{70:\overline{5|}} + {}_5p_{70}(e_{75:\overline{5|}}) + {}_{10}p_{70}(e_{80})$

$= 4.66234 + .8808(4.43230) + .8808(.8020)(8.26871)$

$= 14.4074$

B

18 This is a typical problem on population theory, which has been asked many times on the exam. The method of solution is a little unusual, since it relies on knowledge of how to calculate the complete expectation under the constant force assumption.

The average age at death for those now age x, who die between ages x and x+n equals

$$x + \frac{T_x - T_{x+n} - n(l_{x+n})}{l_x - l_{x+n}} \qquad T_x = \int_0^\infty l_{x+t}\,dt$$

The problem asks for the number of years lived after age 60 for those who die between 60 and 80:

$$z = \frac{T_{60} - T_{80} - 20 l_{80}}{l_{60} - l_{80}}$$

If you know the formula for $\mathring{e}_x$, it will allow you to evaluate z:

$$\mathring{e}_x = \int_0^\infty {}_tp_x\,dt = \frac{\int_0^\infty l_{x+t}\,dt}{l_x} = \frac{T_x}{l_x}$$

$$z = \frac{T_{60}/l_{60} - T_{80}/l_{60} - 20(l_{80}/l_{60})}{1 - l_{80}/l_{60}}$$

(next page)

(18) $$Z = \frac{\mathring{e}_{60} - \frac{l_{80}}{l_{60}}(\mathring{e}_{80}) - 20\left(\frac{l_{80}}{l_{60}}\right)}{1 - \frac{l_{80}}{l_{60}}}$$

You can calculate the value of $l_{80}/l_{60} = {}_{20}p_{60}$ under the assumption of a constant force of mortality. The only remaining task is to evaluate $\mathring{e}_x$:

$$\mathring{e}_x = \int_0^\infty {}_tp_x \, dt$$
$$= \int_0^\infty e^{-.10t} \, dt \quad \text{because } {}_tp_x = e^{-t\mu} \text{ (constant force)}$$

$$= \left. \frac{e^{-.10t}}{-.10} \right]_0^\infty$$

NOTE: Based on the chain rule (calculus)
if $g(x) = v(f(x))$
then $g'(x) = v'(f(x)) \cdot f'(x)$

$$= \left(\frac{e^{-\infty}}{-.10} - \frac{e^{-0}}{-.10} \right)$$
$$= 1/.10$$
$$= 10.0$$

$$Z = \frac{10 - {}_{20}p_{60}(10) - 20({}_{20}p_{60})}{1 - {}_{20}p_{60}}$$

$${}_{20}p_{60} = e^{-20(.10)} = .1353$$

$$Z = \frac{10 - .1353(10) - 20(.1353)}{1 - .1353}$$
$$= 6.8696$$

A

solution added 04/12/06

2002

(18) If you like, you can calculate $\mathring{e}_x$ without using any integrals:

$${}_t p_x = e^{-t\mu}$$

$${}_1 p_x = e^{-.10} = .9048 \qquad {}_t p_x = (.9048)^t$$

$$e_x = p_x + {}_2 p_x + {}_3 p_x + \ldots$$
$$= .9048 + (.9048)^2 + (.9048)^3 + \ldots$$

This is a perpetuity immediate, where $v = .9048 \Rightarrow i = 10.52\%$.

$$e_x = \frac{1}{10.52\%} = 9.5083$$

$$\mathring{e}_x \doteq e_x + \tfrac{1}{2} \qquad (\text{true for U.D.D.})$$
$$= 10.0083$$

$$Z = \frac{10.0083 - {}_{20}p_{60}(10.0083) - 20({}_{20}p_{60})}{1 - {}_{20}p_{60}}$$
$$= \frac{10.0083(1-(.9048)^{20}) - 20(.9048)^{20}}{1-(.9048)^{20}}$$
$$= 6.8780 \qquad \text{(A)}$$

The answer is very close to the exact result of 6.8696

2002

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19 This is a typical exam question involving actuarially equivalent optional forms of benefit payment. The key concept is that the present values must all be equal.

The first step is to write expressions for the present values of each option. The key to working this problem is being comfortable writing these expressions based on reversionary annuities

Option

I $\quad PV = 12X\ddot{a}_{65}^{(12)}$

II $\quad PV = 12Y\left[\ddot{a}_{\overline{5}|}^{(12)} + {}_{5|}\ddot{a}_{65}^{(12)}\right]$

III $\quad PV = 12Y\left[\ddot{a}_{65:65}^{(12)}\right] + 12X\left[\ddot{a}_{65}^{(12)} - \ddot{a}_{65:65}^{(12)}\right] + 12PY\left[\ddot{a}_{65}^{(12)} - \ddot{a}_{65:65}^{(12)}\right]$

Unlike other similar problems on past exams, you have values given for the various annuities. You need to calculate the value for $\ddot{a}_{\overline{5}|}^{(12)}$. My preference is to use a monthly interest rate, and to value 60 payments of 1/12.

annual $i = 7\%$ monthly $j \Rightarrow (1+j)^{12} = 1.07 \quad \therefore j = .5654\%$

$\ddot{a}_{\overline{5}|.07}^{(12)} = 1/12\left(\ddot{a}_{\overline{60}|.5654}\right)$

$= 1/12\,(1.005654)(50.7617)$

$= 4.2541$

After you set the present values equal to each other, you have 2 equations in 3 unknowns - not a good sign!

(next page)

(19) Here is what you have so far

$$I = II: \quad 12X\ddot{a}_{65}^{(12)} = 12Y\left[4.2541 + {}_{5|}\ddot{a}_{65}^{(12)}\right]$$

$$I = III: \quad 12X\ddot{a}_{65}^{(12)} = 12Y\left[\ddot{a}_{65:65}^{(12)}\right] + (12X + 12PY)\left[\ddot{a}_{65}^{(12)} - \ddot{a}_{65:65}^{(12)}\right]$$

From the first equation, you can express X in terms of Y. After you substitute in the second equation, you will have Y on both sides, so you can eliminate it, which leaves an equation only in terms of P.

$$12X\ddot{a}_{65}^{(12)} = 12Y\left[4.2541 + {}_{5|}\ddot{a}_{65}^{(12)}\right]$$

$$X = Y\left[4.2541 + 6.0553\right]/10.0833$$

$$= 1.0224Y$$

$$12(1.0224Y)\ddot{a}_{65}^{(12)} = 12Y\left[\ddot{a}_{65:65}^{(12)}\right] + (12(1.0224Y) + 12PY)\left[\ddot{a}_{65}^{(12)} - \ddot{a}_{65:65}^{(12)}\right]$$

$$1.0224\,\ddot{a}_{65}^{(12)} = \ddot{a}_{65:65}^{(12)} + (1.0224 + P)\left[\ddot{a}_{65}^{(12)} - \ddot{a}_{65:65}^{(12)}\right]$$

$$\frac{1.0224\,\ddot{a}_{65}^{(12)} - \ddot{a}_{65:65}^{(12)}}{\ddot{a}_{65}^{(12)} - \ddot{a}_{65:65}^{(12)}} = 1.0224 + P$$

$$\frac{1.0224(10.0833) - 9.5833}{10.0833 - 9.5833} - 1.0224 = P$$

$$P = \frac{.7261}{.5000} - 1.0224$$

$$= 42.97\%$$

B

20 This type of problem was asked on the 2000 and 2001 exam also. You are told that the single decrement tables are linear between ages 41 and 42.

I will assume that the single decrement tables have uniform distribution of decrements. The other two "standard" mortality assumptions are constant force of mortality, and the Balducci assumption. Neither of those gives a linear result for the q_x values over one year of age.

If you are told that the probabilities in the multiple decrement table conform to U.D.D., then you would use one of these formulas:

$$q'^{(1)}_x \doteq \frac{q^{(1)}_x}{1-\frac{1}{2}q^{(2)}_x} \qquad \text{or} \qquad p'^{(1)}_x = \left[p^{(\tau)}_x\right]^{q^{(1)}_x/q^{(\tau)}_x}$$

The corresponding formula based on U.D.D. in the single decrement tables is discussed in Bowers 10.6.3. The formula based on 2 decrements is derived as part of the solution for 2000 # 11:

$$q'^{(1)}_x \doteq \frac{q^{(1)}_x}{1-\frac{1}{2}q'^{(2)}_x}$$ (This formula uses the rate of decrement in the denominator)

(next page)

(20) The problem asks for the number of decrements due to cause 1 at age 41 $= l_{41}^{(T)} q_{41}^{(1)} = d_{41}^{(1)}$. You need to derive the values for various rates and probabilities to calculate that value.

$q_x'^{(1)}[1-\frac{1}{2}q_x'^{(2)}] \doteq q_x^{(1)}$ based on earlier formula U.D.D. in single decrement tables

$l_{41}^{(T)} = p_{40}^{(T)} l_{40}^{(T)}$

$p_{40}^{(T)} = p_{40}'^{(1)} p_{40}'^{(2)} = (1-.05)(1-.10)$

$= .8550$

$l_{41}^{(T)} = .8550(10{,}000)$

$= 8{,}550$

$p_{41}^{(T)} = l_{42}^{(T)} / l_{41}^{(T)}$

$= 7{,}000/8{,}550 = .8187$

$p_{41}^{(T)} = p_{41}'^{(1)} p_{41}'^{(2)} = [1-q_{41}'^{(1)}][1-q_{41}'^{(2)}]$

$1-q_{41}'^{(2)} = p_{41}^{(T)} / [1-q_{41}'^{(1)}]$

$= .8187/[1-.06]$

$= .8710$

$\therefore q_{41}'^{(2)} = .1290$

Finally can use formula at top of page for $q_{41}^{(1)}$:

$q_{41}^{(1)} \doteq q_{41}'^{(1)}[1-\frac{1}{2}(q_{41}'^{(2)})]$

$= .06[1-.5(.1290)]$

$= .0561$

$d_{41}^{(1)} = l_{41}^{(T)} q_{41}^{(1)} = 8{,}550(.0561)$

$= 479.9$

A

21 This is a bit different than some other actuarial equivalence problems. It is unusual to have a constant q_x value at the older ages. The key to the problem is setting up expressions for present values in simplest terms, and then figuring out how to evaluate them.

Form 1: $PV = 100(\ddot{a}_{\overline{10}|.07} + {}_{10|}\ddot{a}_{72})$

Form 2: $PV = X(\ddot{a}_{72:75}) + 1.10X(\ddot{a}_{72} - \ddot{a}_{72:75})$
$\qquad + .5X(\ddot{a}_{75} - \ddot{a}_{72:75})$

It may be difficult to write down the expression for payment form 2 if you are not comfortable with reversionary annuities. Since the payment forms are actuarially equivalent, you can set them equal to each other. First you need to figure out how to evaluate $\ddot{a}_x$ and $\ddot{a}_{xy}$.

$$\ddot{a}_x = 1 + v p_x + v^2 {}_2p_x + \ldots$$
$$= 1 + (1.07)^{-1}(1-.04) + (1.07)^{-2}(1-.04)^2 + \ldots$$
$$= 1 + \frac{.96}{1.07} + \left(\frac{.96}{1.07}\right)^2 + \ldots$$

$$\frac{.96}{1.07}\ddot{a}_x = \frac{.96}{1.07} + \left(\frac{.96}{1.07}\right)^2 + \ldots$$

$$\left(1 - \frac{.96}{1.07}\right)\ddot{a}_x = 1$$

$\ddot{a}_x = 9.7273$ at every age x

(next page)

(21) You should be able to perform similarly with $\ddot{a}_{xy}$

$$\ddot{a}_{xy} = 1 + v\,p_{xy} + v^2\,{}_2p_{xy} + \cdots$$
$$= 1 + \frac{.96}{1.07}(.96) + \left(\frac{.96}{1.07}\right)^2(.96)^2 + \ldots$$

$$\frac{(.96)^2}{1.07}\ddot{a}_{xy} = \frac{(.96)^2}{1.07} + \frac{(.96)^4}{(1.07)^2} + \ldots$$

$$\left[1 - \frac{(.96)^2}{1.07}\right]\ddot{a}_{xy} = 1$$

$\ddot{a}_{xy} = 7.2102$ at every age

Form 1: $\ddot{a}_{\overline{10}|.07} = 1.07(a_{\overline{10}|.07}) = 1.07(7.0236) = 7.5152$

$${}_{10|}\ddot{a}_{72} = v^{10}\,{}_{10}p_{72}(\ddot{a}_{82}) = (1.07)^{-10}(.96)^{10}(9.7273)$$
$$= 3.2875$$
$$PV = 100\,(7.5152 + 3.2875)$$
$$= 1{,}080.27$$

Form 2: $PV = X(\ddot{a}_{72:75}) + 1.10X(\ddot{a}_{72} - \ddot{a}_{72:75})$

$$+ .5X(\ddot{a}_{75} - \ddot{a}_{72:75})$$
$$= X(7.2102) + (1.10X + .5X)(9.7273 - 7.2102)$$
$$= X(7.2102) + 1.6X(2.5170)$$
$$= 11.2375X$$

Now set both present values equal, and solve for X:

$$11.2375X = 1{,}080.27$$
$$X = 96.13$$

C

22 This is a very messy arithmetic problem involving life ensurance. The key to the problem is writing down the death benefits and the premium payments on a time line diagram. Then you have to be very careful in solving for the value of X:

Insurance	200,000	. . .	. . .	. . .	. . .	200,000	. . .	~~X~~	X	→			0
Premium	300	600	1200	2400	4800	0	. . .	4000	0	. . .		0	0
Age	25	26	27	28	29	30	. . .	55	56	. . .		65	66

There are only six premium payments, with an unusual pattern. To solve for X, you can equate the present value of the premium payments to the present value of the insurance benefits:

$$PV \text{ of premiums} = 300 + 600 v p_{25} + 1200 v^2 {}_2p_{25} + 2400 v^3 {}_3p_{25} + 4800 v^4 {}_4p_{25} + 4000 v^{30} {}_{30}p_{25}$$

$$PV \text{ of insurance} = 200{,}000 A^{1}_{25:\overline{30}|} + v^{30} {}_{30}p_{25} (A^{1}_{55:\overline{10}|}) X$$

You are given various factors, which allow you to calculate the present value of premiums. Then you can solve for the value of X.

$$PV \text{ of premiums} = 300 + 600(1.06)^{-1} p_{25} + 1200(1.06)^{-2} p_{25} p_{26} + 2400(1.06)^{-3} p_{25} p_{26} p_{27} + 4800(1.06)^{-4} p_{25} p_{26} p_{27} p_{28} + 4000(1.06)^{-30} p_{25} ({}_{29}p_{26})$$

$$= 300 + 600(1.06)^{-1}(.99877) + \ldots \text{ (lots of arithmetic!!)}$$

$$= 8349.44 = 300 + 565.34 + 1{,}065.33 + 2{,}007.38 + 3{,}782.24 + 629.16$$

(next page)

(22) The factors you are given for the insurance benefits don't quite match the initial expression for the present value. You need to rearrange and redefine so you can evaluate the PV of death benefits:

$$\text{PV of death benefits} = 200{,}000\left(A_{25} - {}_{30}p_{25}v^{30}A_{55}\right) + X\left(v^{30}{}_{30}p_{25}\right)\left(A_{55} - v^{10}{}_{10}p_{55}A_{65}\right)$$

$$= 200(1{,}000)\left(A_{25} - (1.06)^{-30}p_{25}({}_{29}p_{26})A_{55}\right) + X\left[(1.06)^{-30}p_{25}({}_{29}p_{26})A_{55} - (1.06)^{-40}{}_{40}p_{25}A_{65}\right]$$

$$= \text{PV of premiums}$$

$$= 8349.44$$

$$8349.44 = 200\left[1000A_{25} - (1.06)^{-30}p_{25}({}_{29}p_{26})1000A_{55}\right] + X\left[(1.06)^{-30}p_{25}({}_{29}p_{26})1000A_{55} - (1.06)^{-40}{}_{40}p_{25}(1000A_{65})\right]/1000$$

The factors you are given are for $1000A_x$. If you don't allow for the 1000, your answer will be off by 1000!

$$X = \frac{8349.44 - 200\left[1000A_{25} - (1.06)^{-30}p_{25}({}_{29}p_{26})1000A_{55}\right]}{.001\left[(1.06)^{-30}p_{25}({}_{29}p_{26})1000A_{55} - (1.06)^{-40}{}_{40}p_{25}(1000A_{65})\right]}$$

$$= \frac{8349.44 - 200\left[81.6496 - .15729(305.1431)\right]}{.001\left[.15729(305.1431) - .0972(.78766)(439.7965)\right]}$$

$$= \frac{1{,}618.65}{.0143}$$

$$= 113{,}059$$

B

This was definitely a LONG 5 point problem.

23 This sounds like a measurement of mortality problem (or maybe survival models). First, translate the data into a mortality study:

(i) The mortality table conforms to De Moivre's law, with one death at each age.

(ii) Deaths were observed at times 4, 5, and 7. The remaining life was still active at time r.

(iii) The lives were observed from time 3 to r.

(iv) ω corresponds to the last age in the mortality table. The maximum likelihood estimate for ω is 13.67.

Now it is unclear exactly what to do with all this information.

no answer!

24 The key to this problem is twofold:
i) ability to interpret the information you are given,
ii) knowledge of identities between insurances and annuities

First, you can write down expressions for the present value of premiums and present value of the death benefits for the insurance policy. Then you can write down similar expressions for the annuity:

Insurance: PV of premiums $= 300$
PV of death bens $= v^4 {}_4p_{60}(1000A_{64})$

Annuity: PV of premiums $= P$
PV of payments $= \ddot{a}_{\overline{3}|.07} + v^3 {}_3p_{60}\ddot{a}_{63}$

For both the insurance and the annuity, the PV of the premiums equals the PV of the benefits. You need to be able to express the value of A_{64} in terms of $\ddot{a}_{63}$ to be able to evaluate P:

$$A_{64} = 1 - d\ddot{a}_{64}$$
$$\ddot{a}_{64}(vp_{63}) = a_{63}$$

$$A_{64} = \frac{300}{1000v^4 {}_4p_{60}}$$ based on the insurance information
$$= \frac{.3(1.07)^4}{(1-.002)^4} = .3964$$

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(24) Now you can determine a value for $\ddot{a}_{63}$ based on the value of A_{64}, and use that to calculate P:

$$\ddot{a}_{64} = \frac{1-A_{64}}{d}$$
$$= \frac{1-.3964}{iv}$$
$$= \frac{1.07(.6036)}{.07}$$
$$= 9.2264$$

$$\ddot{a}_{63} = 1 + v p_{63}(\ddot{a}_{64})$$
$$= 1 + (1.07)^{-1}(1-.002)(9.2264)$$
$$= 9.6056$$

$$P = \ddot{a}_{\overline{3}|.07} + v^3\,{}_3p_{60}\,\ddot{a}_{63}$$
$$= 1.07(a_{\overline{3}|.07}) + (1.07)^{-3}(.998)^3(9.6056)$$
$$= 10.6021$$

C

25 The key to working this problem is knowing that the mortality table matches De Moivre's law. Under that assumption, you can write down a short identity for a_x:

$l_x = \omega - x$ De Moivre's law - here $\omega = 101$ based on $S(x) = \frac{l_x}{l_0}$

let $n = \omega - x$

$$a_x = \frac{n - \ddot{a}_{\overline{n|}} i}{n \cdot i}$$

In this problem, $\omega = 101$, there is one death at each age, and the mortality table matches De Moivre's law.

$n = 101 - 80 = 21$

$\ddot{a}_{80} = 1 + a_{80}$

$$a_{80} = \frac{21 - 1.05(a_{\overline{21|}.05})}{21(.05)}$$

$$= \frac{7.5378}{1.05}$$

$$= 7.1788$$

$\ddot{a}_{80} = 1 + 7.1788$
$= 8.1788$

B

26 This is an unusual gross premium problem. The definition of the annuity is similar to a "Full cash refund" annuity. For this type of annuity, you must use trial and error to determine the present value that exactly matches the death benefit definition.

$$PV \text{ of gross premiums} = PV \text{ of expenses} + PV \text{ of benefits}$$
$$G = .08G + 6{,}000(a^{(12)}_{\overline{65:\overline{n}|}})$$

In the expression for G, the certain period n should equal (G/6,000). Since you are only given deferred annuity values, you should expand the definition of the certain and life annuity:

$$G = .08G + 6{,}000\left[a^{(12)}_{\overline{n}|.05} + {}_{n|}a^{(12)}_{65}\right]$$

$$G = \frac{6{,}000}{.92}\left[a^{(12)}_{\overline{n}|.05} + {}_{n|}a^{(12)}_{65}\right]$$

Since you are only given three deferred annuity values, you can try the middle value, and see if you need to produce a lower (or higher) present value.

Try n=14. 6000(14)=84,000 for the death benefit, which looks reasonable compared to the answer ranges.

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(26) You need to calculate $a^{(12)}_{\overline{14}|.05}$. I will convert this to a monthly annuity:

$$a^{(12)}_{\overline{14}|.05} = \frac{1}{12}\left(a_{\overline{168}|j}\right) \quad \text{where } 1+j = (1.05)^{1/12} \Rightarrow j = .4074\%$$

$$G = \frac{6000}{.92}\left[\frac{a_{\overline{168}|.4074\%}}{12} + 2.92\right]$$

$$= 85{,}066$$

Comparing this to $6000(14) = 84{,}000$, it appears that the refund period should be slightly greater than 14 years. You should try the definition of the present value based on the 15 year deferred annuity.

Try $n=15$. $6000(15) = 90{,}000$ for the death benefit, which looks reasonable based on the answer ranges.

$$a^{(12)}_{\overline{15}|.05} = \frac{1}{12}\left(a_{\overline{180}|.4074\%}\right)$$

$$G = \frac{6000}{.92}\left[\frac{a_{\overline{180}|.4074\%}}{12} + 2.56\right]$$

$$= 85{,}927$$

Since both calculations fall in the same range, the correct answer is C. You could use interpolation to determine the exact value of the refund period, which is between 14 and 15 years. But that is not necessary, since the answer will be in range C.

27 There are two different methods of solution for this problem. The shortest answer is based on a less theoretical approach.

In general $\mu_x = \left(-\frac{d\, l_x}{dx}\right)\frac{1}{l_x}$

Given $\mu_x^{(1)} = \frac{1}{100-x}$, you can guess that $l_x^{(1)} = 100-x$.

Since $\mu_x^{(2)} = \frac{1}{100-x}$, you also have $l_x^{(2)} = 100-x$.

$l_x^{(1)} - l_{x+1}^{(1)} = d_x^{(1)} = 1$ (These values are for the single decrement table)

$l_0^{(1)} = 100$ $\quad l_{10}^{(1)} = 90$ $\quad l_0^{(2)} = 100 = l_0^{(1)}$

$q_0'^{(1)} = 1/100$ $\quad q_0'^{(2)} = 1/100 = q_0'^{(1)}$

${}_{10}p_0'^{(1)} = 90/100$ $\quad {}_{10}p_0'^{(2)} = .90$

$${}_{10}p_0^{(T)} = {}_{10}p_0'^{(1)} \cdot {}_{10}p_0'^{(2)} = .81$$

$${}_{10}q_0^{(T)} = 1-.81 = .19 = {}_{10}q_0^{(1)} + {}_{10}q_0^{(2)}$$

Since ${}_nq_0'^{(1)} = {}_nq_0'^{(2)}$, it makes sense that ${}_nq_0^{(1)} = {}_nq_0^{(2)}$.
Therefor ${}_{10}q_0^{(1)} = .19/2 = .095$

B

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(27) Another solution technique uses integration formulas to arrive at the same result

$$\mu_x^{(T)} = \mu_x^{(1)} + \mu_x^{(2)} = 2/(100-x)$$

$${}_tp_x^{(T)} = e^{-\int_x^{x+t} \mu_y^{(T)}\,dy}$$

$$= e^{\int_x^{x+t} \left(\frac{2}{y-100}\right)dy}$$

$$= e^{2\int_x^{x+t} (y-100)^{-1}\,dy}$$

$$= e^{\left[2(\log(y-100)\right]_x^{x+t}}$$

$$= e^{2(\log(x+t-100)-\log(x-100))}$$

$$= e^{\log\left(\frac{x+t-100}{x-100}\right)^2}$$

$${}_tp_x^{(T)} = \left(\frac{x+t-100}{x-100}\right)^2$$

$${}_{10}p_0^{(T)} = \left(\frac{-90}{-100}\right)^2$$

$$= .81$$

$${}_{10}q_0^{(T)} = .19 = 1-.81$$

$$= {}_{10}q_0^{(1)} + {}_{10}q_0^{(2)}$$

$$= 2\left({}_{10}q_0^{(1)}\right) \text{ based on } \mu_x^{(1)} = \mu_x^{(2)}$$

$${}_{10}q_0^{(1)} = .095 = .19/2$$

B