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SPRING 2003 EA-1
EXAM SOLUTIONS

Revision History:
04/12/06 Added alternate solution for problem 22

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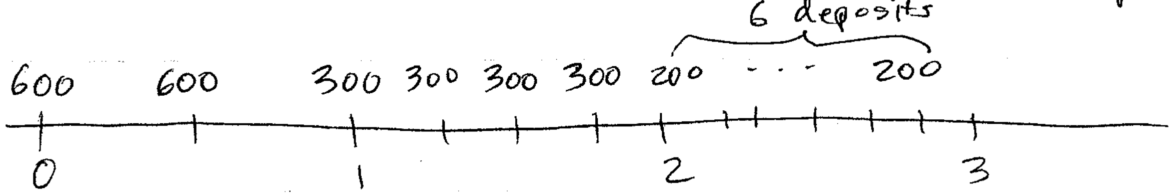
2003 EA-1

1 This is a typical question that has appeared on prior EA-1 exams. based on the information given, you need to calculate the annual effective rate of interest for each year. Then determine the rate of interest for the frequency of deposits in that year.

First step is to write down the series of deposits on a time line diagram. There are \$1200 deposited each year:



In year 1, $d^{(12)} = 6\%$ $\quad 1+i = \left(1 - \frac{.06}{12}\right)^{-12} = 1.0620$

Accumulated value of deposits at time 1:

$600\, s_{\overline{2}|j}$ where $(1+j)^2 = 1.0620$ $\quad j = 3.05\%$

$= 1255.52$

I prefer to convert the interest rate to match the payment period. Alternatively, you get the same results with $1200\, s^{(2)}_{\overline{1}|6.20\%}$.

I also try to calculate all annuities using the formula for annuity immediate. The reason is that I use the HP-12C calculator. I don't want to switch the calculator between end of the year and beginning of the year annuities – that tends to produce too many careless errors

(next page)

(1) Continued

In year 2 $i^{(3)} = 8\%$ $\quad 1+i = \left(1+\frac{.08}{3}\right)^3 = 1.0822$

Accumulated value of all deposits at time 2:

$1255.52\,(1.0822) + 300\,s_{\overline{4}|k}$ where $1+k = (1.0822)^{\frac{1}{4}} - 1 \Rightarrow k = 1.99\%$

$= 1255.52\,(1.0822) + 300\,(4.2034)$

$= 2619.68$

In year 3 $\delta = 7\%$ $\quad 1+i = e^{.07} = 1.0725$

Accumulated value of all deposits at time 3:

$2619.68\,(1.0725) + 200\,s_{\overline{6}|m}$ where $1+m = (1.0725)^{1/6} - 1 \Rightarrow m = 1.17\%$

$= 2619.68\,(1.0725) + 200\,(6.2513)$

$= 4{,}059.89$

D

2 You can easily derive the formula for a continuous perpetuity from regular annuity formulas

$a_{\overline{n}|i} = \frac{1-v^n}{i}$ $\quad \bar{a}_{\overline{n}|i} = \frac{1-v^n}{\delta}$ $\quad \bar{a}_{\overline{\infty}|i} = \lim_{n\to\infty} \bar{a}_{\overline{n}|i}$

$= \frac{1-v^\infty}{\delta}$

PV of all payments: $10{,}000/\delta$ $\quad = 1/\delta$

$$\frac{10{,}000}{\delta} = (\text{PV of A}) + (\text{PV of B}) + (\text{PV of C})$$

$$= 10{,}000\left[\bar{a}_{\overline{x}|} + v^x \bar{a}_{\overline{y}|} + v^{x+y}\left(\frac{1}{\delta}\right)\right]$$

$B = C/2$

$$10{,}000\, v^x \bar{a}_{\overline{y}|} = \frac{1}{2}\left(10{,}000 \frac{v^{x+y}}{\delta}\right)$$

$$v^x\left(\frac{1-v^y}{\delta}\right) = \frac{1}{2}\, v^x \frac{(v^y)}{\delta}$$

$$1 - v^y = \frac{1}{2} v^y$$

$$v^y = 2/3$$

D

3 This is a messy loan problem. You have geometrically increasing payments for 30 years. The first step is to solve for the amount of the initial payment. You should write a time line diagram that shows all the payments:

Date	1-03	1-04	1-05	....	1-23	...	1-33
Time	0	1	2	....	20	...	30
Pmt		Z	$Z(1.03)^1$	...	$Z(1.03)^{19}$	...	$Z(1.03)^{29}$

You can calculate the principal repaid in the first 20 payments as the difference between the initial loan of 10,000, and the outstanding balance after the 20th payment. The interest paid will equal the sum of the first 20 payments less the principal repaid.

$$10{,}000 = v^1 Z + v^2 Z(1.03) + \dots + v^{30} Z(1.03)^{29}$$

$$= vZ\left[1 + \frac{1.03}{1.075} + \dots + \left(\frac{1.03}{1.075}\right)^{29}\right]$$

$$= \frac{Z}{1.075}\ \ddot{a}_{\overline{30}|j} \quad \text{where } 1+j = \frac{1.075}{1.03} = 1.0437$$

$$= \frac{Z}{1.075}(1.0437)\, a_{\overline{30}|4.378}$$

$$Z = \frac{10{,}000\,(1.075)}{16.5431\,(1.0437)} = 622.62$$

(next page)

(3) continued

$$X = \sum_1^{20} \text{All payments} - \sum_1^{20} \text{Principal repaid}$$

$$\sum_1^{20} \text{Principal repaid} = 10{,}000 - (\text{O/S loan balance after } 20^{th} \text{ pmt})$$

$$\text{O/S loan balance}_{20} = \text{PV of remaining payments}$$
$$= vZ(1.03)^{20} + \dots + v^{10}Z(1.03)^{29}$$
$$= Z(1.03)^{19}\left[\frac{1.03}{1.075} + \dots + \left(\frac{1.03}{1.075}\right)^{10}\right]$$
$$= 622.62(1.03)^{19}(a_{\overline{10}|}4.37\%) = 622.62(1.7535)(7.9640)$$
$$= 8{,}694.79$$

$$\sum_1^{20} \text{Principal repaid} = 10{,}000 - 8694.79$$
$$= 1{,}305.21$$

$$\sum_1^{20} \text{All payments} = Z[1 + (1.03)^1 + \dots + (1.03)^{19}]$$
$$= 622.62\, s_{\overline{20}|}.03$$
$$= 622.62\,(26.8704)$$
$$= 16{,}730.03$$

$$X = \sum_1^{20} \text{All payments} - \sum_1^{20} \text{Principal repaid}$$
$$= 16{,}730.03 - 1{,}305.21$$
$$\approx 15{,}425$$

C

4 This is a typical problem involving a mortgage. There is no significant difference between problems involving loans or mortgages – they are solved in essentially the same way.

You need to determine the outstanding balance of the mortgage after the 120^{th} payment. Then you can recalculate the monthly payment based on the new interest rate and payment period.

You are given annual effective rates of interest. You should determine the equivalent monthly interest rate, since the payments are made on a monthly basis.

Let X be the initial mortgage payment

O/S balance at time 0 $= 300{,}000 = X\, a_{\overline{360}|j}$ where $(1+j)^{12} = 1.075$

$j = .6045\%$

O/S balance at time 120 $= X a_{\overline{240}|j} - 2{,}000$

$$= 300{,}000\left(\frac{a_{\overline{240}|.6045}}{a_{\overline{360}|.6045}}\right) - 2{,}000$$

$$= 300{,}000\left(\frac{127.25}{147.42}\right) - 2{,}000$$

$$= 256{,}954$$

New term : 20 years annual rate 7.0%

$P\, a_{\overline{240}|k} = 256{,}954$ $(1+k)^{12} = 1.07 \Rightarrow k = .5654\%$

$$P = \frac{256{,}954}{a_{\overline{240}|.5654\%}}$$

$$= 1959.13$$

C

2003

5 Problems on time weighted and dollar weighted returns require that you write the market values and cash flows on a time line diagram. The time ~~weighted~~ return uses ratios of the market values (at the cash flow dates), and measures the change in market value between the cash flows

You need to determine the time weighted returns separately for Smith and Jones. You are given market values prior to the cash flows, and you need to calculate the market values after the cash flows.

SMITH

	1/1	4/1	7/1	10/1	12/31
(before) MV_B	2,500,000		2,800,000 1,000,000 transfer		4,180,000
(after) MV_A	2,500,000		3,800,000		4,180,000

The time weighted return calculation uses ratios of the market values. The numerators are at the top of the diagram, and denominators are at the bottom:

$$1+i_S = \frac{2,800,000}{2,500,000}\left(\frac{4,180,000}{3,800,000}\right) = 1.232 \Rightarrow i_S = 23.2\%$$

JONES

	1/1	4/1	7/1	10/1	12/31
(before) MV_B	2,500,000 1,500,000 contrib		4,500,000 -1,000,000 transfer		3,500,000
(after) MV_A	4,000,000		3,500,000		3,500,000

(next page)

(5) continued

$$1+i_J = \frac{4,500,000}{4,000,000}\left(\frac{3,500,000}{3,500,000}\right) = 1.125 \Rightarrow i_J = 12.5\%$$

$$X = \text{average} = \tfrac{1}{2}(23.2\% + 12.5\%) = 17.85\%$$

The dollar weighted return calculation is based on the total assets for Smith and Jones

	1/1	4/1	7/1	10/1	12/31
MV_B	5,000,000				
TOTAL	1,500,000 contrib				
MV_A (after)	6,500,000				7,680,000

The dollar weighted return is not dependent on the intervening market values. You need to solve for the return that allows the initial market value plus the cash flows to accumulate to the ending market value.

$$6,500,000(1+Y) = 7,680,000$$
$$Y = 18.15\%$$

$$Y - X = 18.15\% - 17.85\% = .30\%$$ (D)

6 Under the Balducci assumption, the inverse of the l_x values follows a straight line between integer ages:

$$\frac{1}{l_{x+t}} = \frac{1-t}{l_x} + \frac{t}{l_{x+1}}$$

Use the formula to derive values for $l_{60.75}$ and $l_{61.75}$. The difference between the two values is the number of deaths between those ages:

$$\frac{1}{l_{60.75}} = \frac{.25}{l_{60}} + \frac{.75}{l_{61}} = \frac{.25}{10{,}000} + \frac{.75}{7{,}500} = .000125$$

$$l_{60.75} = 1/.000125$$
$$= 8{,}000$$

$$\frac{1}{l_{61.75}} = \frac{.25}{l_{61}} + \frac{.75}{l_{62}} = \frac{.25}{7{,}500} + \frac{.75}{5{,}625} = .000167$$

$$l_{61.75} = 1/.000167$$
$$= 6{,}000$$

$$X = l_{60.75} - l_{61.75}$$
$$= 2{,}000$$

B

7 This is similar to other recent problems on stationary population theory. The other problems gave you information about $\mathring{e}_y$, the complete expectation of life.

For this problem, the key is knowing the formula for the average age at death of those who die between y and y+n:

$$\text{Avg age at death} = y + \frac{T_y - T_{y+n} - n(l_{y+n})}{l_y - l_{y+n}}$$

The problem asks for the number of deaths before age 25 each year, which is $l_0 - l_{25}$. Based on the information given, you have the following:

$T_0 = 9800$

$\underbrace{l_{25} - l_\omega}_{\text{Deaths} > 25} = 4\underbrace{(l_0 - l_{25})}_{\text{Deaths} < 25}$ ω is omega → last age in table

$$\text{AVG age at death for} > 25 = 66 = 25 + \frac{T_{25} - T_\omega - (\omega - 25) l_\omega}{l_{25} - l_\omega}$$

$$= 25 + \frac{T_{25}}{l_{25}} \Rightarrow \frac{T_{25}}{l_{25}} = 41$$

$$\text{AVG age at death for} < 25 = 16 = 0 + \frac{T_0 - T_{25} - 25 l_{25}}{l_0 - l_{25}}$$

(7) continued

$l_{25} - l_{\omega} = l_{25} = 4(l_0 - l_{25})$

$.25 l_{25} = l_0 - l_{25}$

$l_0 = 1.25 l_{25}$

AVG age at death for <26 $= 16 = \dfrac{0 + 9800 - T_{25} - 25 l_{25}}{1.25 l_{25} - l_{25}}$

$4 l_{25} = 9800 - (41 l_{25}) - 25 l_{25}$ used $T_{25} = 41 l_{25}$

$70 l_{25} = 9800$

$l_{25} = 140$

$l_0 = 1.25(140)$

$= 175$

$l_0 - l_{25} = 175 - 140$

$= 35$

E

8 This is a typical actuarial equivalence problem. The key idea is that the present values of actuarially equivalent benefits are equal. In this problem, you have to calculate monthly annuities certain.

$PV \text{ of } A = 12(1{,}000)\ddot{a}_{60}^{(12)}$
$PV \text{ of } B = 12(X)\ddot{a}_{\overline{60:\overline{5|}}}^{(12)}$
$PV \text{ of } C = 12[(X+300)\ddot{a}_{\overline{3|}}^{(12)} + Y[v^3\ddot{a}_{\overline{2|}}^{(12)} + 5|\ddot{a}_{60}^{(12)}]]$

$\ddot{a}_{\overline{n|}i}^{(12)} = \dfrac{\ddot{a}_{\overline{n|}i}}{d^{(12)}} = \dfrac{1}{12}(\ddot{a}_{\overline{12n|}})$ at $i^{(12)}$

Equivalent monthly rate $i^{(12)} = (1.07)^{\frac{1}{12}} - 1 = .565\%$

$\ddot{a}_{\overline{2|}}^{(12)} = (1/12)(1.00565)\,a_{\overline{24|}.565}$ Always use annuity immediate
$= 1.8759$ since I'm using the HP-12C
$\ddot{a}_{\overline{3|}}^{(12)} = (1/12)(1.00565)a_{\overline{36|}.565} = 2.7228$
$\ddot{a}_{\overline{5|}}^{(12)} = (1/12)(1.00565)a_{\overline{60|}.565} = 4.2541$

Using A and B, you can solve for the value of X:

$$12(1000)\ddot{a}_{60}^{(12)} = 12(X)(\ddot{a}_{\overline{5|}}^{(12)} + 5|\ddot{a}_{60}^{(12)})$$
$$X = \frac{1000\,\ddot{a}_{60}^{(12)}}{\ddot{a}_{\overline{5|}}^{(12)} + 5|\ddot{a}_{60}^{(12)}}$$
$$= \frac{1000(11.53)}{4.2541 + 7.35}$$
$$= 993.62$$

(8) Continued

Now you can use A and C to solve for the value of Y:

$$12(1000)\ddot{a}_{60}^{(12)} = 12\left[(X+300)\,\ddot{a}_{\overline{37}|}^{(12)} + Y\left(v^{3}\ddot{a}_{\overline{27}|}^{(12)} + {}_{5|}\ddot{a}_{60}^{(12)}\right)\right]$$

$$Y = \frac{1000\,\ddot{a}_{60}^{(12)} - (X+300)\,\ddot{a}_{\overline{3}|}^{(12)}}{v^{3}\ddot{a}_{\overline{27}|}^{(12)} + {}_{5|}\ddot{a}_{60}^{(12)}}$$

$$= \frac{1000(11.53) - (300+993.62)(2.7228)}{(1.07)^{-3}(1.8759) + 7.35}$$

$$= \frac{8{,}007.75}{8.8813}$$

$$= 901.65$$

(D)

The problem also gave you the value of ${}_{3|}\ddot{a}_{60}^{(12)}$. If you set up the formulas incorrectly, this would allow you to work the problem and get the wrong answer!

9 There are several pieces of information you need to work this problem. First is an understanding of the multiple decrement table information given. Next, you need to read the death benefit definition carefully. Finally you must be able to write the formula for the one year term cost.

You are given 1000 participants at 01/01/03. Based on the total probability of survival, there are 990 participants left at 01/01/04. The missing piece of information is the probability of mortality at age 61. (All of the 01/01/04 participants are age 61)

$$p_{61}^{(total)} = 1 - q_{61}^{(mort)} - q_{61}^{(dis)} \quad \text{standard definition - multiple decrements}$$
$$.985 = 1 - q_{61}^{(mort)} - .011$$
$$q_{61}^{(mort)} = .004$$

The death benefits for 2004 will be those paid from 01/01/05 through 12/01/07. This is a three year annuity certain, payable monthly:

$$\ddot{a}_{\overline{3}|.07}^{(12)} = \frac{1}{12}\ddot{a}_{\overline{36}|j} \quad \text{where } (1+j)^{12} = 1.07 \Rightarrow j = .565\%$$
$$= \frac{1}{12}(1.00565)\, a_{\overline{36}|.565\%}$$
$$= 2.7228$$

The one year term cost for 2004 death benefits is the present value of all payments attributable to the 2004 expected exits. You are told that this is valued at 1-1-04.

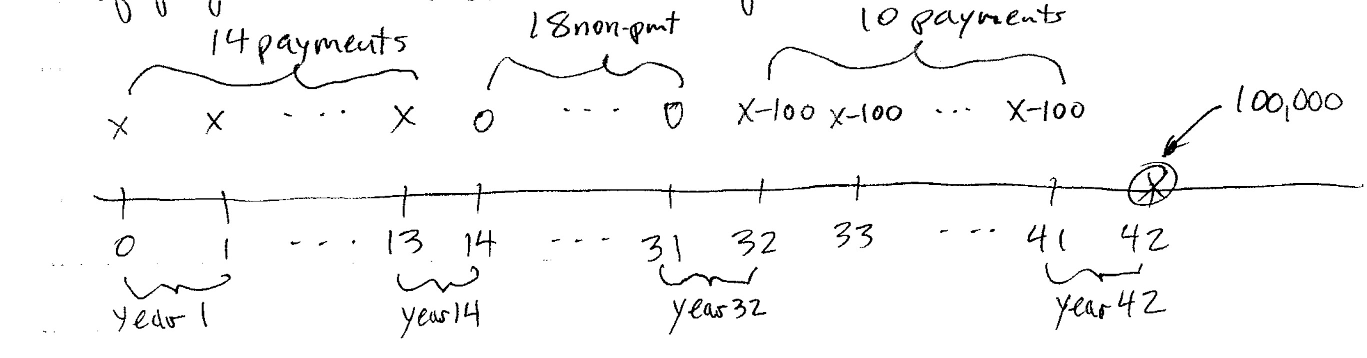
(next page)

(9) Continued

$$X = v^1 \left(q_{61}^{(mort)}\right)(990)(\text{PV of death ben at 12-31-04})$$
$$= (1.07)^{-1}(.004)(990)\left(12{,}000\ \ddot{a}^{(12)}_{\overline{3}|.07}\right)$$
$$= .9346(.004)(990)(12{,}000)(2.7228)$$
$$= 120{,}923$$

(D)

10 This is a fairly typical question involving non-level payments. The key is to carefully write down the series of payments on a time line diagram:



There are several ways to write the accumulated value

$$100{,}000 = X\ddot{s}_{\overline{42}|.07} - 100\ddot{s}_{\overline{10}|.07} - X(\ddot{s}_{\overline{18}|.07})(1.07)^{10}$$
$$= X\ddot{s}_{\overline{14}|.07}(1.07)^{28} + (X-100)\ddot{s}_{\overline{10}|.07}$$

The second one might be slightly easier to understand. Both expressions will generate the same value for X. The first step is to express everything using immediate annuities, since I use the HP-12C calculator

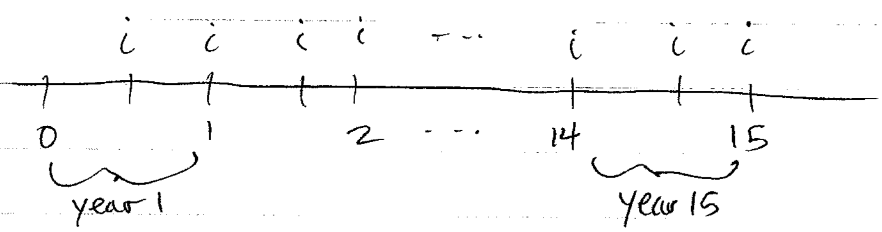
$$100{,}000 = X s_{\overline{14}|.07}(1.07)^{29} + (X-100)(s_{\overline{10}|.07})(1.07)$$
$$X\left(s_{\overline{14}|.07}(1.07)^{29} + s_{\overline{10}|.07}(1.07)\right) = 100{,}000 + 100(s_{\overline{10}|.07})(1.07)$$
$$X = \frac{100{,}000 + 100(s_{\overline{10}|.07})(1.07)}{s_{\overline{14}|.07}(1.07)^{29} + s_{\overline{10}|.07}(1.07)}$$
$$= \frac{100{,}000 + 100(13.8164)(1.07)}{22.5505(7.1143) + 13.8164(1.07)}$$
$$= \frac{101{,}478}{175.2136}$$
$$= 579.17$$

C

2003

11 The main point of the problem is your ability to interpret the information given. You need to assume the amount of the initial deposit. The interest credit is $i/2$ each six months, since you are told the nominal rate is i compounded semi-annually.

I'll assume the initial deposit is 2. The interest earned every 6 months is $2(i/2) = i$:



These interest payments are accumulated in a fund that earns 6% per annum (effective). I'll convert the interest rate to match the payment period (semi-annual). That will make it easier to calculate the accumulated fund value:

$$(1+j)^2 = 1.06 \Rightarrow j = 2.956\%$$

After 15 years, accumulated value of all payments:

$$\underbrace{2 + i\,(s_{\overline{30}|2.956\%})}_{\text{fund interest earned}} = 2(1.0756)^{15}$$

7.56% yield rate

$$i = \frac{2(1.0756)^{15} - 2}{s_{\overline{30}|2.956\%}}$$

$$= 3.9675 / 47.240$$

$$= 8.40\%$$

(c)

12 If you understand sinking funds, this is an inspection problem. The higher the return in the sinking fund, the lower the total annual payment:

$$P = .08(10{,}000) + \frac{10{,}000}{s_{\overline{10}|j}}$$

II will be the highest payment, due to lowest sinking fund rate

III will be next highest, sinking fund earns 8%

I will be identical to III : $\frac{1}{a_{\overline{n}|i}} = i + \frac{1}{s_{\overline{n}|i}}$

IV will be lowest payment, due to highest sinking fund rate

IV < I = III < II

E

You can also calculate the payment amounts

I. $10{,}000 / a_{\overline{10}|.08} = 1490.29$

II. $.08(10{,}000) + 10{,}000 / s_{\overline{10}|.06} = 1558.68$

III $.08(10{,}000) + 10{,}000 / s_{\overline{10}|.08} = 1490.29$

IV $.08(10{,}000) + 10{,}000 / s_{\overline{10}|.12} = 1369.84$

The key idea with a sinking fund is that you pay the interest separately each year. Additional payments are made into the sinking fund, which accumulate to the desired principal amount (usually at different interest rate).

13 The key to this problem is knowing how to write down the amortization schedule for the loan:

$P = \text{Loan} / a_{\overline{20}|i}$

$\text{Loan} = P\, a_{\overline{20}|i}$

Interest in 1^{st} payment $= i(\text{Loan}) = P(1-v^{20})$

Principal in 1^{st} payment $= P - \text{interest} = Pv^{20}$ $(1+20=21)$

Principal in 13^{th} payment $= Pv^{8}$ $(13+8=21)$

Principal in 18^{th} payment $= Pv^{3}$ $(18+3=21)$

$Pv^8 = 54.40$

$Pv^3 = 70.09$

$(1+i)^5 = 1.2884$

$1+i = 1.051990$

$i = 5.199\%$

$P = 70.09 / v^3$

$= 81.60$

Total payments $= 20P$

Total principal $= P\,a_{\overline{20}|i}$ = original loan amount

Total interest $= 20P - P\,a_{\overline{20}|i}$

$= 81.60(20 - 12.2546)$

$= 632.02$

B

14 This is a theoretical problem, which tests your ability to handle continuous annuities, the force of interest, and the force of mortality. Most of the work is simply to solve for the interest rate

$\bar{a}_{\overline{n}|i} = \dfrac{1-v^n}{\delta}$ $\qquad X = {}_{20}p_{20}(1 - {}_{10}p_{40})$

$\bar{a}_{\overline{20}|i} = 1.4\, \bar{a}_{\overline{10}|i}$

$\dfrac{1-v^{20}}{\delta} = 1.4\left(\dfrac{1-v^{10}}{\delta}\right)$

$1-v^{20} = 1.4 - 1.4v^{10}$

Let $z = v^{10}$

$1-z^2 = 1.4 - 1.4z$

$z^2 - 1.4z + .4 = 0$ $\qquad$ Now use quadratic formula to solve for z

$az^2 + bz + c = 0 \qquad \Rightarrow z = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{1.4 \pm \sqrt{(1.4)^2 - 4(1)(.4)}}{2(1)}$

$= (1.4 \pm \sqrt{.36})/2$

$z = v^{10} = 1.0$ or $.40$

The value of 1.0 for v^{10} implies $i=0$. This results in both the force of interest and force of mortality $=0$, and $X=0$. This result seems illogical, and is a spurious result of using the quadratic equation.

(14) Continued

Assume $v^{10} = .40$. Based the equality of the force of interest and the force of mortality, plus the fact that you have a constant force at each age, you have ${}_{10}p_y = .40$ for all ages. You can also derive this:

$v^{10} = .40$

$v = .9124$

$1+i = 1.09596$

$e^{\delta} = 1+i = 1.09596$ — δ is the force of interest

$\delta = \ln(1.09596)$

$= .09163$

$\mu = \delta = .09163$ — μ is the force of mortality

${}_t p_y = e^{-t\mu}$ — constant force of mortality

${}_{10}p_y = e^{-10(.09163)}$

$= .40$ — as we expected!

${}_{20}p_y = (.40)^2$

$= .16$

$X = {}_{20}p_{20}(1 - {}_{10}p_{40})$

$= .16(1 - .40)$

$= .096$

D

15 This is a typical problem on the exam. You need to carefully write down the series of payments for the annuities on a time line diagram. Then you need to calculate the interest rate based on the payment frequency, so you can easily determine the present value of each annuity.

The present value of annuity C equals the sum of the present values for A and B. Start with annuity A:

A									
		1,000		1,100		1,200 ...			1,900
	1/1/03	12/31							
time	0	1	2	3	4	5	...	18	19
payment		1		2		3	...		10

The present value of annuity A is the sum of two annuities. One is a level annuity of 10 payments of 1,000 and the other is an increasing annuity of 9 payments that starts at 100. The annual rate of interest is 7%, but the payments are made every other year. The interest rate that matches the payment period is 14.49%:

$$1+j = (1.07)^2 = 1.1449 \quad \Rightarrow \quad j = 14.49\%$$

$$PV \text{ of } A = (1.07)^{-1}\left[1{,}000\, \ddot{a}_{\overline{10}|j} + 100\, (Ia)_{\overline{9}|j}\right]$$

$$= (1.07)^{-1}\left[1{,}000\, \ddot{a}_{\overline{10}|.1449} + 100\left(\frac{\ddot{a}_{\overline{9}|.1449} - 9(1.1449)^{-9}}{.1449}\right)\right]$$

$$= (1.07)^{-1}\left[1{,}000(1.1449)\, a_{\overline{10}|.1449} + \frac{100}{.1449}\left(1.1449(a_{\overline{9}|.1449}) - 9(1.1449)^{-9}\right)\right]$$

$$= .9346\left[1{,}000(1.1449)(5.1179) + \frac{100}{.1449}\left(1.1449(4.8595) - 9(.2959)\right)\right]$$

(15) continued

$PV \text{ of } A = .9346\,(5{,}859 + 2{,}002)$
$= 7{,}347$

Now handle annuity B in a similar fashion. This time there are 10 payments that "increase" geometrically:

(B)

	*						
		$2{,}000$		$2{,}000(.95)^1$	$\ldots$		$2{,}000(.95)^9$
	1/1/03	12/31					
time	0	1	2	3	$\ldots$	18	19
pmt		1		2	$\ldots$		10

$$PV \text{ of } B = (1.07)^{-1}\left[2{,}000\left(1 + \frac{.95}{(1.07)^2} + \cdots + \frac{(.95)^9}{(1.07)^{18}}\right)\right]$$

$$= (1.07)^{-1}\, 2{,}000\, \ddot{a}_{\overline{10}|k} \quad \text{where} \quad 1+k = \frac{(1.07)^2}{.95} = 1.2052$$

$$= (1.07)^{-1}\, 2{,}000\, (1.2052)\, a_{\overline{10}|.2052}$$

$$= .9346(2{,}000)(1.2052)(4.1201)$$

$$= 9{,}281$$

C is a perpetuity due with payments of X. The present value of C equals the sum of the other present values:

$PV \text{ of } C = X/d = X(1.07)/.07 = 7{,}347 + 9{,}281$

$X = 16{,}628(.07)/1.07$

$= 1{,}087.83$

(B)

16 The key idea in this problem is that ~~the~~ price of the bond should take into account any risk of default:

No default risk : Price $= Fra_{\overline{n}|i} + Cv^n$
Default risk q : Price $= (Fra_{\overline{n}|i} + Cv^n)(1-q)$

F is the face value of the bond
r is the coupon rate
n is the number of coupons
i is the yield rate
C is the redemption value

$$980 = \left(1000(10\%)a_{\overline{1}|.07} + 1000(1.07)^{-1}\right)(1-q)$$
$$= \left(1000(10\%)(1.07)^{-1} + 1000(1.07)^{-1}\right)(1-q)$$
$$1-q = \frac{980}{1000(1.07)^{-1}(10\%+1)}$$
$$q = 1 - \frac{980(1.07)}{1100}$$
$$= 4.67\%$$

E

17 There are two ways to work this problem. The easier approach is to write the annuity formulas:

$$\begin{aligned}
&10{,}000\, a_{65:62:\overline{2}|} && \text{pay 10,000 for 2 years - both alive}\\
&+5{,}000\,(a_{65:\overline{2}|} - a_{65:62:\overline{2}|}) && \text{pay 5,000 for 2 years - age 65 alive}\\
&+5{,}000\,(a_{62:\overline{2}|} - a_{65:62:\overline{2}|}) && \text{pay 5,000 for 2 years - age 62 alive}\\
\hline
&5{,}000\, a_{65:\overline{2}|} + 5{,}000\, a_{62:\overline{2}|}
\end{aligned}$$

The resulting annuities are easily calculated. You can see that the sum of these annuities provides the payments described while both annuitants are alive, or if either one dies.

$$\begin{aligned}
PV &= 5{,}000\,(v p_{65} + v^2\, {}_2p_{65}) + 5{,}000\,(v p_{62} + v^2\, {}_2p_{62})\\
&= 5{,}000\left(\frac{.965}{1.07}\left(1+\frac{.960}{1.07}\right)\right) + 5{,}000\left(\frac{.98}{1.07}\left(1+\frac{.975}{1.07}\right)\right)\\
&= 5{,}000\,(1.7110 + 1.7505)\\
&= 17{,}307 \qquad \text{B}
\end{aligned}$$

It is possible to write down all the probabilities for the payments each year. This requires you to carefully enumerate all the cases for survival and death:

$$\begin{aligned}
&v\,[10{,}000\, p_{65} p_{62} + 5{,}000\,(p_{65} q_{62} + q_{65} p_{62})]\\
&+v^2\,[10{,}000\, {}_2p_{65}\, {}_2p_{62} + 5{,}000\,({}_2p_{65}(q_{62} + p_{62} q_{63}) + (q_{65} + p_{65} q_{66})\, {}_2p_{62}]\\
&= .9346\,[10{,}000(.9457) + 5{,}000(.0536)]\\
&\quad + .8734\,[10{,}000(.8852) + 5{,}000(.1124)]\\
&= 17{,}311
\end{aligned}$$

18 This problem tests your understanding of the relationship between single decrement tables and multiple decrement tables. The key point is that retirements occur at the beginning of the year. You should determine the number of retirements first, and then allow for the effect of the other decrements.

To get the deaths in 2003, you need to construct the multiple decrement table value for $q_{54}^{(d)}$. You are told that the single decrement tables have uniform distribution of decrements. Based on the two decrements at age 54, you have Bowers 10.6.3 formula

$$q_x^{(d)} \doteq q_x^{\prime(d)}\left(1-\tfrac{1}{2}\left(q_x^{\prime(\omega)}\right)\right)$$
$$= .0044\,(1-.5(.04))$$
$$= .004312$$
$$q_x^{(\omega)} \doteq q_x^{\prime(\omega)}\left(1-\tfrac{1}{2}\left(q_x^{\prime(d)}\right)\right)$$
$$= .04\,(1-.5(.0044))$$
$$= .039912$$

Deaths at age 54 in 2003 $= 1{,}000(.004312) = 4.312$

$$p_{54}^{(T)} = 1 - q_{54}^{(d)} - q_{54}^{(\omega)}$$
$$= 1 - .004312 - .039912$$
$$= .955776$$

This also equals $(1-q_{54}^{\prime(d)})(1-q_{54}^{\prime(\omega)})$

$$l_{55}^{(T)} = 1000\,(.955776)$$
$$= 955.776$$

(next page)

(18) continued

The key to the problem is to allow for retirements at age 55 before the deaths. Since the retirements occur at the beginning of the year, they are not exposed to the force of mortality.

Retirements at 55 = 955.776(.15) $\quad$ Note $q_{55}^{(r)} = q'^{(r)}_{55}$

= 143.37

Note that you should not use the same formula as before to determine $q_{55}^{(r)}$. Since the decrement occurs at the beginning of the year, $q_{55}^{(r)} = q'^{(r)}_{55} = .15$.

Deaths at 55 = $(955.776 - 143.37)\, q_{55}^{(d)}$

= 812,410 (.0049) $\quad$ Note $q_{55}^{(d)} = q'^{(d)}_{55}$

= 3.981

We could have used the multiple decrement formula for death and withdrawal to calculate $q_{55}^{(d)}$ based on $q'^{(d)}_{55}$ and $q'^{(w)}_{55}$. But the withdrawal rate is zero, so $q_{55}^{(d)} = q'^{(d)}_{55}$.

Deaths in 2003 and 2004 = 4.312 + 3.981

= 8.293 $\quad$ C

19 This is a typical problem involving joint and survivor annuities. The key to the problem is reading the annuity description carefully, and writing down the correct expression. With actuarially equivalent annuities, the present values are equal:

Option 1: $PV = 10{,}000\,\ddot{a}_{65}$
$= 10{,}000(10.0426)$

Option 2: $PV = .75Y\,\ddot{a}_{65} + .75Y\,\ddot{a}_{62} + (1-2(.75))\ddot{a}_{65:62}$
$= Y[.75(10.0426) + .75(10.6974) - .5(8.7060)]$

$$Y = \frac{10{,}000(10.0426)}{.75(10.0426+10.6974) - .5(8.7060)}$$
$$= \frac{100{,}426}{11.2020}$$
$$= 8965.01$$

D

The formula used for the J+S annuity which pays \$K to the survivors, and \$1 while both are alive is
$K\ddot{a}_x + K\ddot{a}_y + (1-2K)\ddot{a}_{xy}$

Another way to get the same result is using the idea of reversionary annuities. The reversionary annuity $K(\ddot{a}_x - \ddot{a}_{xy})$ pays K to a life age x after the death of a life age y. Here is the same present value as above

$$K(\ddot{a}_x - \ddot{a}_{xy}) + K(\ddot{a}_y - \ddot{a}_{xy}) + 1(\ddot{a}_{xy})$$

20 This is a typical problem involving loan amortization schedules. The question itself is a bit unusual. You need to determine the series of interest payments for a loan at 8%, then calculate the present value of interest payments using 12% interest.

Initial loan $1{,}000\, a_{\overline{25}|.08}$

Amortization schedule

Payment	Interest paid	Principal paid	O/S Loan
1	$1000(1-v^{25})$	$1000v^{26}$	$1{,}000\, a_{\overline{24}\|.08}$
⋮	⋮	⋮	⋮
25	$1000(1-v^{1})$	$1000v^{1}$	$1000\, a_{\overline{0}\|.08}$

PV of interest payments at 12%:

$$X = (1.12)^{-1}\left(1000-1000(1.08)^{-25}\right) + (1.12)^{-2}\left(1000-1000(1.08)^{-24}\right) + \dots + (1.12)^{-25}\left(1000-1000(1.08)^{-1}\right)$$

$$= 1000\, a_{\overline{25}|.12} - 1000\left[(1.12)^{-1}(1.08)^{-25} + (1.12)^{-2}(1.08)^{-24} + \dots + (1.12)^{-25}(1.08)^{-1}\right]$$

$$= 1000\left[7.8431 - (1.08)^{-26}\left[(1.12)^{-1}(1.08)^{1} + (1.12)^{-2}(1.08)^{2} + \dots + (1.12)^{-25}(1.08)^{25}\right]\right]$$

The key to working this problem is seeing how to get the exponents for the 8% and 12% terms to have the same absolute value. The key thing to notice was that the sum of the exponents in the prior series was always 26.

$$X = 1000\left[7.8431 - (1.08)^{-26}\left[\frac{1.08}{1.12} + \left(\frac{1.08}{1.12}\right)^2 + \dots + \left(\frac{1.08}{1.12}\right)^{25}\right]\right]$$

$$= 1000\left[7.8431 - (1.08)^{-26}\, a_{\overline{25}|j}\right] \quad \text{where } 1+j = 1.12/1.08 = 1.03704$$

$$= 1000\left[7.8431 - .1352\,(16.1231)\right]$$

$$= 5{,}663.27$$

C

21 This is a rare exam problem on determination of yield rates. You must calculate the original price of the bond, and the final sale price after 10 years:

$P = Fr a_{\overline{n}|i} + Cv^n$ annual rate 9%

Original bond price semi-annual $j \Rightarrow (1+j)^2 = 1.09$

$P = .04(1000)a_{\overline{30}|4.403\%} + 1000(1.04403)^{-30}$ $j = 4.403\%$

$= 40(16.4763) + 1000(.2745)$

$= 933.59$

Sale price after 5 years annual rate 8.25%

$Q = .04(1000)a_{\overline{20}|k} + 1{,}000(1+k)^{-20}$ Semi-annual $k \Rightarrow (1+k)^2 = 1.0825$

$= 40\, a_{\overline{20}|4.043\%} + 1{,}000(1.04043)^{-20}$ $K = 4.043\%$

$= 40(13.5384) + 1000(.4526)$ confusingly similar digits to j!

$= 994.14$

To measure Smith's rate of return, equate the original bond price to the cash received:

$933.59 = 40\, a_{\overline{10}|m} + 994.14(1+m)^{-10}$ note: $(1+m)^2 = 1+X$

There are now three ways to solve for m:

1) Easiest is to test the answer ranges
2) Next easiest is to let your calculator solve for the yield rate
3) Hardest is to iterate and solve for the yield rate

(21) continued

(1) Test answer ranges

First, it should be clear that Smith's yield is greater than 9.0%. If he had sold the bond at a price that yields 9.0% to the buyer, then Smith's yield would equal 9.0%. Since the buyer of the bond is earning only 8.25%, Smith must earn more than 9.0% on the sale.

You have three possible answer choices, so you should test the yields of 9.5% and 9.9%. That will allow you to identify the correct answer range:

Annual yield of 9.5% ⇒ semi-annual yield of 4.64%

$40\, a_{\overline{10}|4.64\%} + 994.14(1.0464)^{-10} = 945.81$

Annual yield of 9.9% ⇒ semi-annual yield of 4.83%

$40\, a_{\overline{10}|4.83\%} + 994.14\,(1.0483)^{-10} = 931.48$

Actual yield is between 9.5% and 9.9% per annum (D)

(2) Use calculator

This is easy, if you know exactly how to do it. The trick is knowing how to enter the date information. On the HP-12C, it "grunted" for about 15 seconds, and returned 4.796% for the yield. You then need to calculate the equivalent annual rate $= (1.04796)^2 - 1 = 9.82\%$ (D)

(next page)

(21) continued (2)

(3) Use iteration

Iteration is the least desirable approach. In some cases, the solution will converge slowly. That means it takes too much time to get the desired result. Note that this is a 3 point question, and it shouldn't take too long to solve it.

In the worst case, your iteration technique might actually fail to converge. Then you'll have to try a different formula. There are many possibilities:

$$993.59 = 40a_{\overline{10}|} + 994.14v^{10}$$
$$= 40(1-v^{10})/i + 994.14v^{10}$$

You can rearrange the terms to iterate on either i or v^{10}:

$$v^{10} = \frac{933.59 - 40(1-v^{10})/i}{994.14}$$

$$i = 40(1-v^{10}) / [993.59 - 994.14v^{10}]$$

Using the second formula, we'll make an educated guess of 9.5% to start. The semi-annual equivalent is $4.64\% = (1.095)^{1/2} - 1$:

$i_0 = 4.64\%$ $\quad i_1 = 4.83\%$ $\quad i_2 = 4.801\%$ $\quad i_3 = 4.806\%$ $\quad i_4 = 4.805\%$

At this point you should feel safe that you are closing in on the final result. Annual yield $= (1.04805)^2 - 1 = 9.84\%$ (D)

22 This is a very short problem which tests your knowledge of mortality table formulas. Unlike other similar exam questions, you should not use this formula:

$$q_x^{(d)} \doteq q_x'^{(d)}\left(1 - \tfrac{1}{2}\left(q_x'^{(w)}\right)\right)$$

This formula is based on the single decrement tables both having uniform distribution of decrements. But this condition is not met here, since the single decrement withdrawal table has a constant force of mortality.

You should derive the appropriate formula for $q_x^{(d)}$ based on the general formula:

$$q_x^{(1)} = \int_0^1 {}_tp_x^{(T)}\, \mu_{x+t}^{(1)}\, dt$$

$$q_x^{(d)} = \int_0^1 {}_tp_x^{(T)}\, \mu_{x+t}^{(d)}\, dt \qquad {}_tp_x^{(T)} = {}_tp_x'^{(d)}\, {}_tp_x'^{(w)}$$

$$= \int_0^1 {}_tp_x'^{(d)}\, \mu_{x+t}^{(d)}\, ({}_tp_x'^{(w)}\, dt)$$

Since the single decrement table for death has UDD, you can use the fact that ${}_tp_x\, \mu_{x+t} = q_x$. This is shown in Bowers 3.6.1.

$$q_x^{(d)} = \int_0^1 q_x'^{(d)}\, {}_tp_x'^{(w)}\, dt$$

$$= q_x'^{(d)} \int_0^1 {}_tp_x'^{(w)}\, dt$$

Since $\mu_x^{(w)} = .20$, you can replace ${}_tp_x'^{(w)}$ with $\left[p_x'^{(w)}\right]^t = e^{-.2t}$

(next page)

(22) continued

$$q_x^{(d)} = q_x^{\prime(d)} \int_0^1 e^{-.2t}\,dt$$

$$= q_x^{\prime(d)} \left.\frac{e^{-.2t}}{-.2}\right]_0^1$$

$$= q_x^{\prime(d)} \left[\frac{e^{-.2}}{-.2} - \frac{e^{0}}{-.2} \right]$$

$$= .03 \left[\frac{.8187}{-.2} - \frac{1}{-.2} \right]$$

$$= .03 \left(\frac{1-.8187}{.2} \right)$$

$$= .02719$$

(B)

The wrong way to work this problem is using the formula that assumes UDD in both single decrement tables. It is surprising that this produces the correct result:

$$q_x^{(d)} \doteq q_x^{\prime(d)} \left[1 - \tfrac{1}{2} q_x^{\prime(w)}\right]$$ (WRONG!)

$$p_x^{\prime(w)} = e^{-\mu} = e^{-.20} = .8187$$ withdrawal has constant force

$$q_x^{\prime(w)} = .1813$$

$$q_x^{(d)} = .03\left[1 - \tfrac{1}{2}(.1813)\right]$$
$$= .0273$$

(B)

Note - this the wrong answer, but it is in the correct range!

Solution added 04/12/06

(22) There is one formula that is exact for both of the assumptions, constant force of decrement and uniform distribution of decrements.

$${}_tp_x'^{(1)} = \left[{}_tp_x^{(T)}\right]^{\frac{q_x^{(1)}}{q_x^{(T)}}}$$

See Bowers 10.5.10 and 10.5.12

$$1-q_x'^{(d)} = \left[p_x^{(T)}\right]^{\frac{q_x^{(d)}}{q_x^{(T)}}}$$

$$\log(1-.03) = \frac{q_x^{(d)}}{q_x^{(T)}} \log(p_x^{(T)})$$

$$p_x^{(T)} = p_x'^{(d)} p_x'^{(w)}$$

$$\mu_x^{(w)} = .20 \Rightarrow {}_tp_x'^{(w)} = e^{-.2t} \Rightarrow p_x'^{(w)} = .8187$$

$$p_x'^{(d)} = 1-.03 \qquad = .97$$

$$p_x^{(T)} = .97(.8187)$$
$$= .7942$$

$$\log(.97) = \frac{q_x^{(d)}}{1-.7942} \log(.7942)$$

$$q_x^{(d)} = (1-.7942)\frac{\log(.97)}{\log(.7942)}$$
$$= .2058(.1322)$$
$$= .0272$$

B

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23 This is a messy actuarial equivalence problem. The key idea is writing down the formula for the "refund" death benefit correctly, and finding a way to simplify it. There is a lot of work required to handle the 10 year certain and life form as well:

$$PV = 100\,(\ddot{a}_{\overline{10}|.07} + v^{10}\,{}_{10}p_x\,\ddot{a}_{x+10})$$
$$= Z\ddot{a}_x + v^1 q_x(11Z) + v^2 p_x q_{x+1}(10Z) + \dots + v^{11}\,{}_{10}p_x\, q_{x+10}(Z)$$

You can set up a formula for $\ddot{a}_{x+10}$ using the standard recursive formula between successive ages:

$$v p_x \ddot{a}_{x+1} = \ddot{a}_x - 1.0$$
$$v p_{x+9}\ddot{a}_{x+10} = \ddot{a}_{x+9} - 1.0$$
$$\ddot{a}_{x+9} = 1 + v p_{x+9}\ddot{a}_{x+10}$$
$$\ddot{a}_{x+8} = 1 + v p_{x+8}\ddot{a}_{x+9}$$
$$= 1 + v p_{x+8}(1 + v p_{x+9}\ddot{a}_{x+10})$$
$$= 1 + v p_{x+8} + v^2\,{}_2p_{x+8}\,\ddot{a}_{x+10}$$
$$\ddot{a}_{x+7} = 1 + v p_{x+7} + v^2\,{}_2p_{x+7} + v^3\,{}_3p_{x+7}\,\ddot{a}_{x+10}$$
$$\vdots$$
$$\ddot{a}_x = 1 + v p_x + v^2\,{}_2p_x + \dots + v^{10}\,{}_{10}p_x\,\ddot{a}_{x+10}$$
$$v^{10}\,{}_{10}p_x\,\ddot{a}_{x+10} = \ddot{a}_x - (1 + v p_x + v^2\,{}_2p_x + \dots + v^9\,{}_9p_x)$$

Since you need $v^{10}\,{}_{10}p_x\,\ddot{a}_{x+10}$, just simplify the right hand side!

$$= \ddot{a}_x - \left(1 + \frac{.98}{1.07} + \left(\frac{.98}{1.07}\right)^2 + \dots + \left(\frac{.98}{1.07}\right)^9\right)$$

$$= 9.3 - \ddot{a}_{\overline{10}|j} \quad \text{where } 1+j = 1.07/.98 = 1.0918$$
$$= 9.3 - (1.0918)a_{\overline{10}|.0918}$$
$$= 2.3493$$

(23) continued

$PV = 100\,(\ddot{a}_{\overline{10}|.07} + v^{10}\,{}_{10}p_x\,\ddot{a}_{x+10})$

$= 100\,(1.07\,(a_{\overline{10}|.07}) + 2.3493)$

$= 986.45$

Now you need to simplify the summation for the "refund" death benefit:

$PV = Z\ddot{a}_x$ + refund death benefit

$PV\,death = .02Z[(1.07)^{-1}(11) + (1.07)^{-2}(.98)^{1}(10) + \ldots + (1.07)^{-11}(.98)^{10}(1)]$

$= \frac{.02Z}{.98}\left[\left(\frac{.98}{1.07}\right)(11) + \left(\frac{.98}{1.07}\right)^2(10) + \ldots + \left(\frac{.98}{1.07}\right)^{11}(1)\right]$

$= \frac{.02Z}{.98}\left(Da_{\overline{11}|j}\right)$ where $1+j = 1.07/.98 = 1.0918$

Remember the formula for $Da_{\overline{n}|i} = \frac{n - a_{\overline{n}|i}}{i}$

Total $PV = Z\ddot{a}_x + \frac{.02Z}{.98}\;\frac{(11 - a_{\overline{11}|.0918})}{.0918}$

$= Z(9.3 + .0204\,(11 - 6.7465)/.0918)$

$986.45 = Z(9.3 + .9452)$

$Z = 96.28$

(D)

Another approach would be to write the 10 year certain and life annuity differently:

$PV = 100\,(\ddot{a}_{\overline{10}|.07} + \ddot{a}_x - \ddot{a}_{x:\overline{10}|})$

$\ddot{a}_{x:\overline{10}|} = 1 + v^1 p_x + v^2\,{}_2p_x + \ldots + v^9\,{}_9p_x$

$= 1 + (.98/1.07) + \ldots + (.98/1.07)^9$

$= \ddot{a}_{\overline{10}|.0918}$

$= 6.3661 = 1.0918\,(a_{\overline{10}|.0918})$

This quickly leads to the same result shown at the top of this page

24 This is another problem that requires you to use single decrement tables to calculate values for a multiple decrement table. Since the single decrement tables have uniform distribution of decrements over each year of age, you use this formula to derive the multiple decrement probabilities:

$$q_x^{(2)} \doteq q_x'^{(2)}\left(1-\tfrac{1}{2}q_x'^{(1)}\right)$$

The problem asks for ${}_3q_{60}^{(2)} = q_{60}^{(2)} + p_{60}^{(T)} q_{61}^{(2)} + {}_2p_{60}^{(T)} q_{62}^{(2)}$. You need to calculate several items to "fill in the blanks" in the data given.

$$p_{60}^{(T)} = 1-q_{60}^{(T)} = .92 = 1-.08$$
$$l_{60}^{(T)} = l_{61}^{(T)} / p_{60}^{(T)} = 1875 = 1725/.92$$
$$d_{60}^{(T)} = l_{60}^{(T)} - l_{61}^{(T)} = 150 = 1875-1725$$
$$d_{60}^{(2)} = d_{60}^{(T)} - d_{60}^{(1)} = 30 = 150-120$$
$$q_{60}^{(2)} = d_{60}^{(2)} / l_{60}^{(T)} = .0160 = 30/1875$$

For $q_{61}^{(2)}$, you'll use the single decrement q's in the formula above:

$$q_{61}^{(2)} \doteq q_{61}'^{(2)}\left(1-\tfrac{1}{2}(q_{61}'^{(1)})\right)$$
$$= .10\,(1-.5(.20))$$
$$= .090$$
$$p_{61}^{(T)} = p_{61}'^{(1)} \cdot p_{61}'^{(2)} = (1-q_{61}'^{(1)})(1-q_{61}'^{(2)})$$
$$= .72 = (1-.20)(1-.10)$$
$$l_{62}^{(T)} = l_{61}^{(T)} p_{61}^{(T)} = 1242 = 1725(.72)$$
$$q_{62}^{(2)} = d_{62}^{(2)} / l_{62}^{(T)} = .0620 = 77/1242$$

$${}_3q_{60}^{(2)} = .0160 + .92(.090) + .72(.92)(.0620)$$
$$= .1399$$

D

25 This is an interesting variation on multiple decrement problems. The key point is the assumption that retirements occur in the middle of the year. This is not the same as the assumption of UDD, since you are not exposed to the force of retirement throughout the year.

The participant is age 65 at 01/01/2003, but you should assume a retirement rate of .50 at age 65.5. This is confusing, since most people would expect everyone to retire at exact age 65.

The one year term cost for the death benefit represents the present value of the expected exits due to death for the year. This would normally be written as

$v^1 q_{65}^{(d)}\, 10{,}000$ note: multiple decrement q, not q'

You are given the single decrement $q_{65}^{\prime(d)} = .04$, and told that this table has a constant force of mortality within each year of age. As shown in Bowers 3.6.1, this produces the following relationship

$_tp'_x = e^{-t\mu} = (p'_x)^t$

$_tp'^{(d)}_{65} = (.96)^t$

The value of $q_{65}^{(d)}$ must reflect the probability of survival to age 66. The should allow for mortality only up to age 65.5, then surviving the retirement decrement at 65.5, and then mortality only from age 65.5 to age 66.

(25) continued

As described above, you have

$$q_{65}^{(d)} = {}_{.5}q'^{(d)}_{65} + (1 - {}_{.5}q'^{(d)}_{65})(1 - q'^{(r)}_{65})({}_{.5}q'^{(d)}_{65.5})$$

$${}_{.5}p'^{(d)}_{65} = (.96)^{.5} = .9798 = {}_{.5}p'^{(d)}_{65.5}$$
$${}_{.5}q'^{(d)}_{65} = 1 - .9798 = .0202$$

$$q_{65}^{(d)} = .0202 + (1-.0202)(1-.50)(.0202)$$
$$= .0301$$

$$Y = v\, q'^{(d)}_{65}\, 10{,}000$$
$$= (1.07)^{-1}(.0301)\,10{,}000$$
$$= 281.33$$

A

If you compare this problem to #22 on this exam, you'll notice we did not use the integral formula to calculate $q_{65}^{(d)}$. The reason is that a life age 65 is not subject to the force of retirement throughout the year of age. As stated in the problem, you assume that the retirement decrement occurs at age 65.5.

26 This problem is handled easily if you understand the concept of reversionary annuities, as well as how to write the value of a joint and last survivor annuity. As stated, you can write down a formula for the present value:

$$\ddot{a}_{\overline{xy}:\overline{20|}} + {}_{20|}\ddot{a}_{xy}$$

The first term is a joint and last survivor annuity that pays as long as either x or y is alive, but not beyond 20 years. The second term is a joint life annuity that pays as long as both x and y are alive, but no payments are made for the first 20 years.

$$\ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}$$

$$\ddot{a}_{\overline{xy}:20} = \ddot{a}_{x:\overline{20|}} + \ddot{a}_{y:\overline{20|}} - \ddot{a}_{xy:\overline{20|}}$$

$$= (\ddot{a}_x - {}_{20|}\ddot{a}_x) + (\ddot{a}_y - {}_{20|}\ddot{a}_y) - (\ddot{a}_{xy} - {}_{20|}\ddot{a}_{xy})$$

$$= (10-3) + (8.4-2.5) - (6.2-1.9)$$

The point of the problem is that you have to re-write the formulas to match the data you are given.

$$PV = \ddot{a}_{\overline{xy}:\overline{20|}} + {}_{20|}\ddot{a}_{xy}$$

$$= 8.6 + 1.9$$

$$= 10.5$$

D

27 This is a potentially confusing identity question. I find it easier to work if I don't look at the answers too carefully:

$1+i = \left(1-\frac{d^{(m)}}{m}\right)^{-m}$

$1+i = \left(1-\frac{d^{(4)}}{4}\right)^{-4}$ First, start with $d^{(4)}$ identity

$1-\frac{d^{(4)}}{4} = (1+i)^{-1/4}$

$\frac{d^{(4)}}{4} = 1-(1+i)^{-1/4}$

$d^{(4)} = 4\left[1-(1+i)^{-1/4}\right]$

Now use the information given to derive a value for $(1+i)^{-1/4}$:

$99 = 100(1+i)^{-5/12}$

$(1+i)^{-5/12} = 99/100$

$(1+i)^{-3/12} = (99/100)^{3/5}$

$d^{(4)} = 4\left[1-(99/100)^{3/5}\right]$ A

28 This is a typical actuarial equivalence problem. In keeping with the de-emphasis on commutation functions, the problem gives you information to work the problem with no knowledge of commutation functions.

The two present values of the normal form and alternative form must be equal. This is the basic idea of actuarial equivalence between benefit forms.

PV of normal form: $5000 \frac{D_{65}}{D_{45}} \left(\ddot{a}_{\overline{10}|.05} + \frac{D_{75}}{D_{65}} \ddot{a}_{75} \right)$

PV of alternative form: $Y \frac{D_{55}}{D_{45}} \ddot{a}_{55}$

$$Y = \frac{5000\left(\frac{D_{65}}{D_{45}}\right)\left(\ddot{a}_{\overline{10}|.05} + (D_{75}/D_{65})\ddot{a}_{75}\right)}{\left(\frac{D_{55}}{D_{45}}\right)\ddot{a}_{55}}$$

$$= \frac{500(v^{20}\,{}_{20}p_{45})\left[\ddot{a}_{\overline{10}|.05} + (v^{10}\,{}_{10}p_{65})\ddot{a}_{75}\right]}{(v^{10}\,{}_{10}p_{45})\ddot{a}_{55}}$$

$$= \frac{5000\cancel{(v^{10}\,{}_{10}p_{45})}(v^{10}\,{}_{10}p_{55})\left[\ddot{a}_{\overline{10}|.05} + (v^{10}\,{}_{10}p_{65})\ddot{a}_{75}\right]}{\cancel{(v^{10}\,{}_{10}p_{45})}\ddot{a}_{55}}$$

$$= \frac{5{,}000(1.05)^{-10}(.91)\left[1.05(7.7217) + (1.05)^{-10}(.88)16.5\right]}{17.2}$$

$$= \frac{5{,}000(.6139)(.91)(17.0218)}{17.2}$$

$= 2{,}764$

B