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SPRING 2006 EA-1 EXAM SOLUTIONS

2006 EA-1 Solutions

1 This is a typical exam question on loans. The key to working it is to quickly write down a few lines of the amortization schedule for the loan. Then you can use the relationship between the principal and interest in the 11th loan payment.

Assume the loan payment is X. The initial amount of the loan is $X a_{\overline{20}|i}$. Here is the loan amortization schedule:

Year	Payment	Interest	Principal	O/S Loan
		i(Payment)	X − interest	
0				$X a_{\overline{20}\|i}$
1	X	$X(1-v^{20})$	Xv^{20}	$X a_{\overline{19}\|i}$
2	X	$X(1-v^{19})$	Xv^{19}	$X a_{\overline{18}\|i}$
⋮	⋮	⋮	⋮	⋮
11	X	$X(1-v^{10})$	Xv^{10}	$X a_{\overline{9}\|i}$

Difference is 9 years

$$X(1-v^{10}) = Xv^{10} = 100$$
$$X - Xv^{10} = Xv^{10}$$
$$1 = 2v^{10}$$
$$v = .933 \Rightarrow i = 7.1773\%$$

$$X = 100/v^{10}$$
$$= 200$$
$$X a_{\overline{20}|i} = 200(a_{\overline{20}|7.1773\%})$$
$$= 200(\ddot{a}_{\overline{20}|7.1773\%})/1.071773$$
$$= 2{,}090$$

D

2006

2 This is a typical problem on time weighted and dollar weighted investment returns. The key to working these problems is ~~to~~ write down the cash flows on a time-line diagram.

You must carefully read the data given in the problem. The fund values you are given are immediately <u>prior</u> to the cash flows. In order to calculate the time weighted return, you need to derive the fund values after the cash flows:

Fund Before cash flow	10,000	12,000	X
Contrib/Ben payment	4,000-1,500	2,000-250	
Date	1-1-06	4-1-06	12-31-06
Fund after cash flow	12,500	13,750	

You can use the given dollar weighted return of 10% to solve for the value of X:

$$12{,}500(1.10) + 1{,}750\left(1 + \tfrac{9}{12}(.10)\right) = X$$

$$13{,}750 + 1{,}881.25 = X = 15{,}631.25$$

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(2) For the time-weighted rate of return, you need to calculate the return <u>between</u> the cash flows. This is calculated as the product of several ratios : $\dfrac{\text{Fund value before cash flow}}{\text{Fund value after prior cash flow}}$

I'll re-write the time-line diagram to show how the ratios are constructed:

Fund before cash flow		$\boxed{12,000}$	$\boxed{15,631.25}$
Date	1-1-06	4-1-06	12-31-06
Fund after cash flow	$\boxed{12,500}$	$\boxed{13,750}$	

Time weighted return

$$(1+R) = \left(\frac{12,000}{12,500}\right)\left(\frac{15,631.25}{13,750.00}\right)$$

$$= .9600(1.1368)$$

$$= 1.0913$$

$R = 9.13\%$ A

2006

3 This is a typical exam question on the relationships between annuities, premiums and insurances. Based on the information given, you need to write down the expression for the present value of premiums, which is equal to the present value of the insurance.

I. $10{,}000\, a_{50} = 123{,}325 \quad \Rightarrow \quad a_{50} = 12.3325$

$\ddot{a}_{50} = 13.3325$

II. $P_{50:\overline{20}|} = \dfrac{100{,}000\, A_{50}}{\ddot{a}_{50:\overline{20}|}} = \dfrac{100{,}000\,(1 - d\ddot{a}_{50})}{\ddot{a}_{50:\overline{20}|}} = 3314$

III. $X = \dfrac{100{,}000\, A_{50:\overline{20}|}}{\ddot{a}_{50:\overline{20}|}} = \dfrac{100{,}000\,(1 - d\,\ddot{a}_{50:\overline{20}|})}{\ddot{a}_{50:\overline{20}|}}$

From I and II, you can solve for the value of $\ddot{a}_{50:\overline{20}|}$. You know the value of d at a 5% interest rate, so you can calculate the value of X.

$$100{,}000\,(1 - d\,\ddot{a}_{50|}) = 3314\ \ddot{a}_{50:\overline{20}|}$$

$$\begin{aligned}\ddot{a}_{50:\overline{20}|} &= 100{,}000(1 - iv\,(13.3325)/3314 \\ &= (100{,}000/3314)(1 - .05(13.3325)/1.05) \\ &= 11.0175\end{aligned}$$

$$\begin{aligned}X &= 100{,}000\,(1/\ddot{a}_{50:\overline{20}|} - d) \\ &= 100{,}000\,((11.0175)^{-1} - .05/1.05) \\ &= 4{,}314.59\end{aligned}$$

C

4 This is a typical exam problem on actuarial equivalence. In general, when two (or more) optional forms are actuarially equivalent, they have the same present values.

You need to write down expressions for the present value of the life annuity and the joint and survivor annuity. Then you can solve for the value of X.

For the J+S annuity, it is easiest to write down reversionary annuities for each "survivor group":

Smith + Spouse both alive:	$X\,\ddot{a}^{(12)}_{65:60}$
Spouse alive, Smith dead:	$X\left[\ddot{a}^{(12)}_{60} - \ddot{a}^{(12)}_{65:60}\right]$
Smith alive, Spouse dead:	$1000\left[\ddot{a}^{(12)}_{65} - \ddot{a}^{(12)}_{65:60}\right]$
Total PV for J+S annuity	$X\,\ddot{a}^{(12)}_{60} + 1000\left(\ddot{a}^{(12)}_{65} - \ddot{a}^{(12)}_{65:60}\right)$

Now set the present values equal and solve for X:

$$1000\,\ddot{a}^{(12)}_{65} = X\,\ddot{a}^{(12)}_{60} + 1000\left(\ddot{a}^{(12)}_{65} - \ddot{a}^{(12)}_{65:60}\right)$$

$$X = 1000\,\frac{\ddot{a}^{(12)}_{65:60}}{\ddot{a}^{(12)}_{60}}$$

$$= 658.92$$

B

Note: The specific J+S annuity in this problem is described as a pop-up annuity. If the spouse dies first, Smith's benefit is restored to its original amount, as if the J+S was never elected at all!

5 This is a loan problem with an unusual twist. You need to set up the loan amortization schedule so you can identify the interest payments over the life of the loan:

Payment	Interest	Principal	O/S Loan
0			21,000
1	.06(21,000)	100(1)	21,000 - 100(1)
2	.06(21,000 - 100)	100(2)	21,000 - 100(1+2)
3	.06(21,000 - 100(1+2))	100(3)	21,000 - 100(1+2+3)
⋮	⋮	⋮	⋮
20	.06(21,000 - 100(1+2+...+19))	100(20)	

$\Sigma = 100(1+2+3+\dots+20)$

$= 100(20)(21)/2 = 21{,}000$ ✓

$\sum_{t=1}^{n} t = \frac{(n+1)n}{2}$

Split these into 2 pieces to set up PV of interest payments

$$X = .06(21{,}000)(v^1 + v^2 + \dots + v^{20})$$
$$-100(.06)\left[v^2(1) + v^3(1+2) + v^4(1+2+3) + \dots + v^{19}(1+2+\dots+18) + v^{20}(1+2+\dots+18+19)\right]$$

$$= .06(21{,}000)\, a_{\overline{20}|.06} - 6\left[\left(v^2 + 3v^3 + 6v^4 + \dots + 171v^{19} + 190v^{20}\right)\right]$$

$$(1+i)X = .06(21{,}000)\, \ddot{a}_{\overline{20}|.06} - 6\left[v + 3v^2 + 6v^3 + \dots + 190v^{19}\right]$$

$$iX = .06(21{,}000)\left(\ddot{a}_{\overline{20}|} - a_{\overline{20}|}\right) - 6\left[v + 2v^2 + 3v^3 + \dots + 19v^{19} - 190v^{20}\right]$$

$$X = 21{,}000\left((1.06)a_{\overline{20}|} - a_{\overline{20}|}\right) - 100\left[(Ia)_{\overline{19}|.06} - 190v^{20}\right]$$

$$= 21{,}000(.06)\, a_{\overline{20}|.06} - 100\left[\frac{\ddot{a}_{\overline{19}|.06} - 19v^{19}}{.06} - 190v^{20}\right]$$

$$= 14{,}452 - 100\left[\frac{1.06\left(a_{\overline{19}|.06}\right) - 19v^{19}}{.06} - 190v^{20}\right]$$

$$= 14{,}452 - 100(92.46 - 59.24)$$

$$= 11{,}130$$

D

2006

6 This is the first question in MANY years on the idea of a bond amortization schedule. This can be constructed by following the idea of a loan amortization schedule. The key is that you should use the alternate bond price formula:

Book value / Price / Amortized value $= C + (Fr - Ci)a_{\overline{n}|i}$

Redemption value $= C$ Face $= F$ Coupon $= Fr$

Time	Coupon	Interest Paid	Principal Paid	Amort. Value
0				$C+(Fr-Ci)a_{\overline{n}\|i}$
1	Fr	$i\{C+(Fr-Ci)a_{\overline{n}\|i}\}$ $= Ci + (Fr-Ci)(1-v^n)$ $= Fr + (Ci - Fr)v^n$ $= Fr - (Fr - Ci)v^n$	$(Fr-Ci)v^n$	$C+(Fr-Ci)a_{\overline{n-1}\|i}$
2	Fr	$Fr-(Fr-Ci)v^{n-1}$	$(Fr-Ci)v^{n-1}$	$C+(Fr-Ci)a_{\overline{n-2}\|i}$
⋮	⋮	⋮	⋮	⋮
11	Fr	$Fr-(Fr-Ci)v^{n-10}$	$(Fr-Ci)v^{n-10}$	$C+(Fr-Ci)a_{\overline{n-11}\|i}$
⋮	⋮	⋮	⋮	⋮
16	Fr	$Fr-(Fr-Ci)v^{n-15}$	$(Fr-Ci)v^{n-15}$ ← amount bond is written up in 16^{th} year	

20 year bond $\Rightarrow n = 20$

$$(Fr - Ci)v^{20-15} = 150\%\,(Fr-Ci)v^{20-10}$$

$$[1000(.05) - 1000i]v^5 = 1.5(1000(.05) - 1000i)v^{10}$$

$$v^5 = 1.5v^{10} \Rightarrow (1+i)^5 = 1.5 \Rightarrow i = 8.4472\%$$

$$P = C + (Fr - Ci)a_{\overline{20}|i} = 1000 + [1000(.05) - 1000(.084472)]a_{\overline{20}|8.4472}$$

$$= 1000 - 34.472(9.4999)$$

$$= 672.52$$

C

2006

7 The key to this problem is understanding how select and ultimate mortality tables work. The first step is to write an expression for Z:

$$Z = d_{[53]+2} - d_{[53]+1}$$
$$= (l_{[53]+2} - l_{[53]+3}) - (l_{[53]+1} - l_{[53]+2})$$

All of the l_x terms involve a select life age 53. The probabilities for this life are all on the row in the table where x is 53. Now you can write expressions for each of the $l_{[53]+t}$ values based on $l_{[53]}$:

$$l_{[53]+1} = l_{[53]}(p_{[53]}) = l_{[53]}(1-q_{[53]})$$
$$l_{[53]+2} = (l_{[53]+1})(p_{[53]+1}) = l_{[53]}(1-q_{[53]})(1-q_{[53]+1})$$
$$l_{[53]+3} = (l_{[53]+2})(p_{[53]+2}) = l_{[53]}(1-q_{[53]})(1-q_{[53]+1})(1-q_{[53]+2})$$

Based on the data given, there is only a three year select period. That means $p_{[53]+3}$ is equal to p_{56}. You can use that information to calculate l_{66}:

$$l_{56} = l_{55}(p_{55}) = 13{,}200(1-.130) = 11{,}484$$

Now you can calculate the prior $l_{[53]+t}$ values:

$$l_{[53]+3} = (l_{[53]+2})[1-q_{[53]+2}] \Rightarrow l_{[53]+2} = l_{56}/(1-q_{[53]+2})$$
$$= 11{,}484/(1-.120)$$
$$= 13{,}050$$

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(7) $l_{[53]+2} = (l_{[53]+1})(1-q_{[53]+1}) \Rightarrow l_{[53]+1} = (l_{[53]+2})/(1-q_{[53]+1})$

$$= 13{,}050/(1-.100)$$

$$= 14{,}500$$

Now you can calculate the value of Z:

$$Z = (l_{[53]+2} - l_{56}) - (l_{[53]+1} - l_{[53]+2})$$

$$= (13{,}050 - 11{,}484) - (14{,}500 - 13{,}050)$$

$$= 1{,}566 - 1{,}450$$

$$= 116$$

A

8 This is a commonly asked exam question on stationary population ~~theory~~. The key to working the problem is knowing the formula for the average age at death for those age X who die between ages X and X+n:

$$X + \left(\frac{T_X - T_{X+n} - n(l_{X+n})}{l_X - l_{X+n}} \right) = 55.567$$

In this problem, $X=50$ and $n=10$. The number of deaths between ages 50 and 60 = $l_{50} - l_{60}$. Plug the values given for T_{50} and T_{60} into the expression above, and you can calculate $l_{50} - l_{60}$:

$$50 + \left(\frac{T_{50} - T_{60} - 10(l_{60})}{l_{50} - l_{60}} \right) = 55.567$$

$$\frac{2{,}020{,}000 - 1{,}200{,}000 - 10(l_{60})}{87{,}500 - l_{60}} = 5.567$$

$$820{,}000 - 10 l_{60} = 5.567(87{,}500 - l_{60})$$

$$\therefore l_{60} = 75{,}093$$

$$\therefore l_{50} - l_{60} = 87{,}500 - 75{,}093$$

$$= 12{,}407$$

B

2006

9) This is a typical question that tests your knowledge of various identities. The key is whether you can write the formula for $(Ia)_{\overline{m}|i}$.

$$(Ia)_{\overline{m}|i} = \frac{\ddot{a}_{\overline{m}|} - mv^m}{i}$$

$$(Ia)_{\overline{3n}|i} = \frac{\ddot{a}_{\overline{3n}|i} - 3n(v^{3n})}{i}$$

$$a_{\overline{n}|i} = \frac{1-v^n}{i}$$

$$7.02358 = \frac{1-.50835}{i}$$

$$i = 7.00\%$$

You can use the calculator to solve for n, which is exactly 10.

$$(Ia)_{\overline{3n}|i} = \frac{\ddot{a}_{\overline{30}|.07} - 30v^{30}}{.07}$$

$$= \frac{(1.07)\,a_{\overline{30}|.07} - 30v^{30}}{.07}$$

$$= \frac{1.07(12.4090) - 30(.1314)}{.07}$$

$$= 133.38$$

D

I always use the HP-12C calculator, and I leave it set to calculate annuity values based on end of year payments. That is why I calculate $\ddot{a}_{\overline{n}|i}$ as $(1+i)a_{\overline{n}|i}$.

10 This is a typical question on probability calculations. The key is knowing the definition of the S(x) function, which is the probability of a life surviving from age zero to age x:

$$S(X) = {}_xp_0 = \ell_x / \ell_0$$

$$Y = ({}_5p_{40} - {}_{10}p_{40})({}_5p_{45} - {}_{10}p_{45})({}_5p_{50} - {}_{10}p_{50})$$

$$= \left[\frac{S(45) - S(50)}{S(40)}\right]\left[\frac{S(50) - S(55)}{S(45)}\right]\left[\frac{S(55) - S(60)}{S(50)}\right]$$

Now you should calculate each of the S(x) values, then evaluate the formula for Y:

$$S(40) = 1 - .005(40) - .0005(40)^2 = .72$$
$$S(45) = \vdots \quad \vdots\ (45) \quad \vdots\ (45) = .67375$$
$$S(50) = \quad (50) \quad (50) = .625$$
$$S(55) = \quad (55) \quad (55) = .57375$$
$$S(60) = 1 - .005(60) - .0005(60)^2 = .52$$

$$Y = \left(\frac{.67375 - .625}{.72}\right)\left(\frac{.625 - .57375}{.67375}\right)\left(\frac{.57375 - .52}{.625}\right)$$
$$= .0677(.076)(.086)$$
$$= .000443$$

B

11 This is a typical exam question on the basic definition of "rates" versus probabilities of decrement. The q' are single decrement table rates, and the q with NO "prime" are the multiple decrement table probabilities.

With a 3 decrement table, you have this relationship:

$$p'^{(T)}_x = [1-q'^{(1)}_x][1-q'^{(2)}_x][1-q'^{(3)}_x] = p'^{(1)}_x \cdot p'^{(2)}_x \cdot p'^{(3)}_x$$

$$p^{(T)}_x = 1-q^{(1)}_x - q^{(2)}_x - q^{(3)}_x$$
$$= p'^{(T)}_x$$

The key idea in this problem is that $q^{(2)}_x$ and $q'^{(2)}_x$ must be equal. The reason is that the $q^{(2)}_x$ occurs at the end of the year.

As a result, you can set up a different relationship between the rates of decrement (for cause 1 and 2) and the probabilities of decrement:

$$(1-q'^{(1)}_x)(1-q'^{(3)}_x) = 1-q^{(1)}_x - q^{(3)}_x \qquad \text{based on survival to EOY}$$
$$(1-.05)(1-.30) = .95(.70) = .665$$

There are 665 survivors of decrements 1 and 3 at the end of the year. There are 19.95 exits ($=.03(665)$) due to cause 2. B

12 This is a typical question on duration calculations. The "Macaulay duration" is the same as the regular duration: $\bar{d} = \left(\sum_{t=1}^{n} t v^t R_t\right) / \left(\sum_{t=1}^{n} v^t R_t\right)$

If a problem asks for the "modified duration", it is simply the (regular duration)/(1+i).

The trick to this question is that you have semi-annual coupons. But the period used in the duration calculation is the number of years.

$$\bar{d} = \frac{\Sigma t \cdot (\text{Present value of payment}_t)}{\Sigma \text{ present value of payment}_t}$$

The bond has 6% annual coupons, paid \$30 every six months. It will be simpler to calculate the duration using the 2.5% semi-annual yield rate:

$$\bar{d} = \frac{.5v(30) + 1.0v^2(30) + \ldots + 9.5v^{19}(30) + 10v^{20}(30) + 10v^{20}(1000)}{v(30) + v^2(30) + \ldots + v^{19}(30) + v^{20}(30) + v^{20}(1000)}$$

$$= \frac{.5(30)\, (Ia)_{\overline{20|}.025} + 10{,}000v^{20}}{30\, a_{\overline{20|}.025} + 1{,}000v^{20}}$$

$$= \frac{15\left[(\ddot{a}_{\overline{20|}.025} - 20v^{20})/i\right] + 10{,}000(1.025)^{-20}}{30\, a_{\overline{20|}.025} + 1{,}000(1.025)^{-20}}$$

$$= \frac{15\left[1.025(15.5892) - 20(.6103)\right]/.025 + 10{,}000(.6103)}{30(15.5892) + 1{,}000(.6103)}$$

$$= (2{,}264.08 + 6{,}102.71)/(467.67 + 610.27)$$

$$= 7.76$$

D

2006

13 This is a very typical exam question on bonds. In general, it does not matter which bond price formula you use in a problem:

$$P = Fr\, a_{\overline{n}|i} + Cv^n \qquad \text{general}$$
$$= (Fr - Ci)\, a_{\overline{n}|i} + C \qquad \text{Alternate}$$
$$= K + (g/i)(C-K) \qquad \text{Makeham's}$$

One key idea in this problem is the meaning of book value. That is the same thing as amortized value, which is the bond price at any point in time based on the original yield rate.

10 year bond ⇒ 20 semi-annual coupons

$$P_0 = 1000\left(\tfrac{.03}{2}\right) a_{\overline{20}|.02} + 1000(1.02)^{-20}$$
$$P_5 = 1000(.015)\, a_{\overline{15}|.02} + 1000(1.02)^{-15} \qquad (\text{Book value}_5)$$
$$= 192.74 + 743.01$$
$$= 935.75$$

For a 5% nominal yield rate (compounded monthly), you need to calculate the bond price based on the equivalent semi-annual interest rate:

$$(1+j)^2 = \left(1 + \frac{.05}{12}\right)^{12} \Rightarrow (1+j)^2 = 1.05116 \Rightarrow 1+j = 1.02526$$

$$\text{new } P_5 = 1000(.015)\, a_{\overline{15}|2.526\%} + 1000(1.02526)^{-15}$$
$$= 185.36 + 687.82$$
$$= 873.19$$
$$\Delta P_5 = 62.57 = |935.75 - 873.19|$$

D

2006

14 This looks like a typical convoluted death benefit problem. At 5 points, it will get messy quite quickly.

$$P = 1000\left({}_{20|}\ddot{a}_{45}\right) + \left(1000\,\ddot{a}_{\overline{5|}.07}\right)A^{1}_{45:\overline{20|}}$$

$$= 1000\left(\ddot{a}_{45} - \ddot{a}_{45:\overline{20|}}\right) + 1000\left(\ddot{a}_{\overline{5|}.07}\right)\left(A_{45:\overline{20|}} - v^{20}\,{}_{20}p_{45}\right)$$

$$= 1000\left(13.1949 - 10.9961\right) + 1000\,\ddot{a}_{\overline{5|}.07}\left(1 - d\,\ddot{a}_{45:\overline{20|}} - (1.07)^{-20}(.8771)\right)$$

$$= 2{,}198.80 \quad + \quad 1000(1.07)(a_{\overline{5|}.07})\left(1 - \frac{.07}{1.07}(10.9961) - .2584(.8771)\right)$$

$$= 2{,}198.80 + 1070(4.1002)\left(1 - .7194 - .2267\right)$$

$$= 2{,}198.80 + 236.78$$

$$= 2{,}435.58$$

B

In retrospect, that seemed more like a 4 point problem.

15 This is a typical exam question on multiple live and joint life annuities. The key to this problem is knowing the formula for $a_{\overline{xyz}}$.

The death benefit pays to the twin children, if both are alive after Smith dies. This is a typical reversionary annuity.

$$P = 1000\, a_{65} + 1000(a_{35:35} - a_{35:35:65})$$

The reversionary annuity is easy to understand. You start with $1000\, a_{35:35}$ and subtract the value of payments while all three are alive.

The problem does not give you the joint life annuity for all three lives. You have to derive its value based on the joint and last survivor annuity.

$$a_{\overline{xy}} = a_x + a_y - a_{xy} \qquad \text{(Two life case)}$$

If you know the formula for the two life case, you can easily expand it to the three life case:

$$\begin{aligned} a_{\overline{xyz}} &= a_{\overline{x:yz}} \qquad \text{(Turn this into 2 life J+S)} \\ &= a_x + a_{\overline{yz}} - a_{x:\overline{yz}} \\ &= a_x + (a_y + a_z - a_{yz}) - (a_{xy} + a_{xz} - a_{xyz}) \\ &= a_x + a_y + a_z - (a_{xy} + a_{xz} + a_{yz}) + a_{xyz} \end{aligned}$$

(15) Now you can derive the value of $a_{35:35:65}$

$$a_{\overline{35:35:65}} = a_{35} + a_{35} + a_{65} - (a_{35:65} + a_{35:65} + a_{35:35}) + a_{35:35:65}$$

$$13.8888 = 2(13.5119) + 8.7004 - (2(8.6451) + 13.1360) + a_{35:35:65}$$

$$8.5908 = a_{35:35:65}$$

Now you can calculate the value of the annuity

$$P = 1000a_{65} + 1000(a_{35:35} - a_{35:35:65})$$

$$= 1000(8.7004 + 13.1360 - 8.5908)$$

$$= 13{,}246$$

B

16 This is a typical problem on actuarial equivalence. The key idea is that, if two benefits are actuarially equivalent, then the present values must be equal. As long as you know what the symbols mean, this is a fairly easy question.

I. $PV = 12(1000)\ddot{a}_{55}^{(12)}$

II. $PV = 12(X)\ddot{a}_{55:\overline{7}|}^{(12)} + 12(X-800)\,{}_{7|}\ddot{a}_{55}^{(12)}$

$= 12\left[800\ddot{a}_{55:\overline{7}|}^{(12)} + (X-800)\ddot{a}_{55:\overline{7}|}^{(12)} + (X-800)\,{}_{7|}\ddot{a}_{55}^{(12)}\right]$

$= 12\left[(X-800)\ddot{a}_{55}^{(12)} + 800\ddot{a}_{55:\overline{7}|}^{(12)}\right]$

Now you can set the present values equal, and solve for the value of X.

$$1000\ddot{a}_{55}^{(12)} = (X-800)\ddot{a}_{55}^{(12)} + 800\ddot{a}_{55:\overline{7}|}^{(12)}$$

$$1800\ddot{a}_{55}^{(12)} = X\ddot{a}_{55}^{(12)} + 800\ddot{a}_{55:\overline{7}|}^{(12)}$$

$$1800 - 800\left(\frac{\ddot{a}_{55:\overline{7}|}^{(12)}}{\ddot{a}_{55}^{(12)}}\right) = X$$

$$1800 - 800(5.5/11.33) = X = 1411.65 \qquad \text{E}$$

17 This is a typical exam question on the force of interest. The key definition for working this problem is $1+i=e^{\delta}$. Since the force of interest changes at March 1, you have a different rate of interest for the first 2 months.

$$1+i=\left[1-\frac{d^{(6)}}{6}\right]^{-6}=e^{\delta}$$

Denote interest from 01/01 to 03/01 as i_1 and from 03/01 to 12/31 as i_2

$$(1+i_1)^{2/12}=e^{.06(2/12)}$$
$$i_1=e^{.060}-1.0$$
$$=6.1837\%$$

$$(1+i_2)^{10/12}=e^{.0635(10/12)}$$
$$i_2=e^{.0635}-1.0$$
$$=6.5559\%$$

Value of Account B at 12/31/06:

$$400(1+i)^{8/12}=400\left[\left(1-\frac{d^{(6)}}{6}\right)^{-6}\right]^{8/12}$$
$$=400\left(1-\frac{d^{(6)}}{6}\right)^{-4}$$

Value of Account A at 12/31/06:

$$100(1+i_1)^{2/12}(1+i_2)^{10/12}+100\left[(1+i_2)^{9/12}+(1+i_2)^{6/12}+(1+i_2)^{3/12}\right]$$
$$=100\left[(1.061837)^{2/12}(1.065559)^{10/12}+(1.065559)^{9/12}+(1.065559)^{6/12}+(1.065559)^{3/12}\right]$$
$$=100\left[(1.0101)(1.0543)+1.0488+1.0323+1.0160\right]$$
$$=416.20$$

$$416.20=400\left[1-d^{(6)}/6\right]^{-4}$$
$$(1.0405)^{-1/4}=1-d^{(6)}/6$$
$$d^{(6)}=5.926\%$$

A

18 This is a typical exam question on duration calculations. The "Macaulay duration" is the same as the regular duration: $\bar{d} = \left(\sum_{t=1}^{n} t v^t R_t\right) / \left(\sum_{t=1}^{n} v^t R_t\right)$

If a problem asks for the "modified duration", it is simply the (regular duration)/(1+i).

The key to this problem is that the bond is purchased at par value. Based on the default assumption that the bond is redeemed at par value, the coupon rate must equal the yield rate:

$$P = Fr(a_{\overline{n}|i}) + Cv^n = (Fr - Ci)a_{\overline{n}|i} + C$$

$$\text{if } P = C \text{ then } Fr = Ci \quad \text{since } F = C = P \Rightarrow r = i$$

$$\bar{d} = D = \frac{1\cdot v\cdot Fr + 2\cdot v^2 Fr + 3\cdot v^3 Fr + \dots + 10v^{10}Fr + 10v^{10}F}{v\cdot Fr + v^2 Fr + v^3 Fr + \dots + v^{10}Fr + v^{10}F}$$

$r = 5\%$

$$D = \frac{.05(v + 2v^2 + 3v^3 + \dots + 10v^{10}) + 10v^{10}}{.05(v + v^2 + v^3 + \dots + v^{10}) + v^{10}}$$

$$= \frac{.05(Ia_{\overline{10}|.05}) + 10v^{10}}{.05(a_{\overline{10}|.05}) + v^{10}} \qquad Ia_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

$$= \frac{.05(\ddot{a}_{\overline{10}|.05} - 10v^{10})/.05 + 10v^{10}}{.05(a_{\overline{10}|.05}) + v^{10}}$$

$$= \frac{1.05(a_{\overline{10}|.05}) - 10v^{10} + 10v^{10}}{.05(a_{\overline{10}|.05}) + v^{10}}$$

$$= 8.1078 = \frac{1.05(7.7217)}{.05(7.7217) + .6139}$$

E

19) This is a typical exam question involving the possible "mortality laws". You need to know the relationships between l_x μ_x and q_x. You also need to know how to write the formula for q_x based on having a constant force of mortality.

$\mu_x = \left(\frac{-dl_x}{dx}\right)\frac{1}{l_x}$ $\qquad Z = {}_{2|2}q_{41} = {}_2p_{41}({}_2q_{43})$

μ_{43} under Table A: $\mu_x = \left(\frac{-d\left[(1000(240-2x)^{.5}\right]}{dx}\right)\frac{1}{1000(240-2x)^{.5}}$

$= \frac{-1000(.5)(-2)(240-2x)^{-.5}}{1000 \quad (240-2x)^{.5}}$

$\mu_x = (240-2x)^{-1}$

$\mu_{43} = (240-86)^{-1} = .006494$

Mortality Table B: constant force $\mu = 2(.006494) = .01299$

${}_tp_x = e^{-t\mu}$ With a constant force of mortality

$Z = (e^{-.01299})^2(1-(e^{-.01299})^2)$

$= (.9871)^2(1-(.9871)^2)$

$= .02498$

E

20 This is NOT a typical question on stationary population theory. This is the first time that the Y_x function has ever been tested on an EA-1 exam!

There are several relationships that you need to know:

$$\mathring{e}_x = T_x / l_x$$

$$Y_x - Y_{x+n} - n(T_{x+n}) = \text{Past lifetime since age } x \text{ of members between ages } x \text{ and } x+n$$

$$X + \frac{Y_x - Y_{x+n} - n(T_{x+n})}{T_x - T_{x+n}} = \text{Average age for members between ages } x \text{ and } x+n$$

$$x(T_x) + 2Y_x = \text{Total lifetime of members at ages } x + \text{older}$$

$$\frac{x(T_x) + 2Y_x}{T_x} = \text{Average age at death for members at ages } x \text{ and older}$$

You are told that there are 40 new entrants each year, with a complete expectation of life equal to 55. The new entrants occur at age 27:

$$T_{27}/l_{27} = \mathring{e}_{27} = 55.0 = T_{27}/40$$
$$T_{27} = 2200 = 40(55.0)$$

(20) The firefighters all retire at age 52, and the stationary population has 850 members:

$$T_{27} - T_{52} = 850 \Rightarrow T_{52} = 1350 = 2200 - 850$$

The average age for the entire stationary population is 38 years:

$$38 = 27 + \frac{Y_{27} - Y_{52} - (52-27)T_{52}}{T_{27} - T_{52}}$$

$$= 27 + \frac{(Y_{27} - Y_{52}) - 25(1350)}{850}$$

$$11(850) = Y_{27} - Y_{52} - 33{,}750$$

$$43{,}100 = Y_{27} - Y_{52}$$

Finally, you can calculate the average age at death for the stationary population. The key point is that you can't simply use the formula on the prior page. You need to subtract off the future lifetime for those above age 52, since they are NOT part of the population:

Avg age at death (Ages 27 to 52)

$$= \frac{27(T_{27}) + 2Y_{27} - (52T_{52} + 2Y_{52})}{T_{27} - T_{52}}$$

$$= \frac{27T_{27} + 2(Y_{27} - Y_{52}) - 52T_{52}}{T_{27} - T_{52}}$$

$$= \frac{27(2200) + 2(43{,}100) - 52(1350)}{850}$$

$$= 88.71 = 75{,}400/850$$

E

21 This question touches on the concept of a yield curve, which has never been tested before. The key idea is that a "spot rate" refers to a rate of return calculated based on a zero coupon bond.

The problem tells you that the following is true:

For a zero coupon bond payable at the end of n years	The rate of interest for the n year period is
$n=1$	$i=2\%$
$n=2$	$i=4\%$
$n=3$	$i=5\%$

For the bond with annual coupons of X, you should discount the coupons and redemption value based on the rates shown above.

$$P = vX + v^2X + v^3X + v^3(P) \qquad \text{(assumes constant } i\text{)}$$

$$5000 = X(1.02)^{-1} + X(1.04)^{-2} + X(1.05)^{-3} + 5000(1.05)^{-3}$$

This expression may look strange, but that is the correct interpretation of how you calculate present values under a yield curve. The key is that the interest rate is defined "for the n year period".

$$X\left[(1.02)^{-1} + (1.04)^{-2} + (1.05)^{-3}\right] = 5000\left(1-(1.05)^{-3}\right)$$

$$X = \frac{5000\,(1-.8638)}{.9804+.9246+.8638}$$

$$= 245.89$$

C

22 This is the second question on the concept of a yield curve, which has never been tested before. The key idea is that the forward rate refers to the annual varying interest rates that produce the same present value as the yield curve rates.

For year n	The forward rate i_n is calculated as
$n=1$	$1+i_1 = 1.02$
$n=2$	$1+i_2 = (1.04)^2/(1+i_1) = 1.0604$
$n=3$	$1+i_3 = (1.05)^3/[(1+i_1)(1+i_2)] = 1.0703$
	$= (1.05)^3/(1.04)^2$

At the end of the second year, the <u>forward</u> rate is the rate for the <u>next</u> year. As shown above, the rate is 7.03%

D

The easy way to get this problem wrong is to think that it asks for the forward rate for the 2nd year.

23 This is an unusual problem involving multiple decrement tables. The unusual aspect is that the problem gives you formulas for ${}_tq'^{(k)}_x$ under 2 different single decrement tables. But the problem does not test you on the value of ${}_tq^{(k)}_x$ for $t<1$. As a result, this is a simpler problem than it usually is on the exam.

There is one key formula you must know to work this problem quickly:

$${}_tp'^{(k)}_x = \left[{}_tp^{(T)}_x\right]^{q^{(k)}_x/q^{(T)}_x}$$

For $t=1$ and $k=1$, you have

$$p'^{(1)}_x = \left[p^{(T)}_x\right]^{q^{(1)}_x/q^{(T)}_x}$$

$$p'^{(1)}_x = \left[p^{(T)}_x\right]^{\frac{q^{(T)}_x - q^{(2)}_x}{q^{(T)}_x}}$$

$${}_tq'^{(1)}_x = .1t \Rightarrow q'^{(1)}_x = .1 \quad p'^{(1)}_x = .9$$

Good news - you don't even care about ${}_tq'^{(2)}_x$ at all

$$.90 = (1-.28)^{\frac{.28-q^{(2)}_x}{.28}}$$

$$\log(.9) = \frac{.28-q^{(2)}_x}{.28}\log(.72)$$

$$.28\,\frac{\log(.90)}{\log(.72)} = .28-q^{(2)}_x$$

$$\begin{aligned} q^{(2)}_x &= .28\left[1-(\log .90)/(\log .72)\right] \\ &= .28\left[1-(-.10536)/(-.32850)\right] \\ &= .1902 \end{aligned}$$

C

24 This is a straightforward question on the relationship between annuities and insurance. You also need to know how to deal with the select and ultimate mortality. There are two identity formulas you will use:

$A_x = 1 - d\ddot{a}_x$

$d\ddot{a}_x = 1 - A_x$

$\ddot{a}_x = \dfrac{1-A_x}{d} = \dfrac{1-.5842}{.06/1.06}$

$= 7.3458$

$\ddot{a}_x = 1 + v p_x \ddot{a}_{x+1}$

Based on select and ultimate:

$\ddot{a}_{[x]} = 1 + v p_{[x]} \ddot{a}_{[x]+1}$

Since the select period is only one year, $\ddot{a}_{x+1} = \ddot{a}_{[x]+1}$. You can use the annuity formulas on the right to solve for the value of $\ddot{a}_{x+1}$, then for $\ddot{a}_{[x]}$.

$\ddot{a}_x = 1 + v p_x \ddot{a}_{x+1}$

$(\ddot{a}_x - 1.0)\left(\dfrac{1+i}{p_x}\right) = \ddot{a}_{x+1} = (7.3458 - 1.0)(1.06/.955)$

$= 7.0435$

$\ddot{a}_{[x]} = 1 + v p_{[x]} \ddot{a}_{[x]+1}$

$= 1 + (1.06)^{-1}(1 - .5q_x)(\ddot{a}_{x+1})$

$= 1 + .9434(1 - .5(.045))(7.0435)$

$= 7.4953$

D

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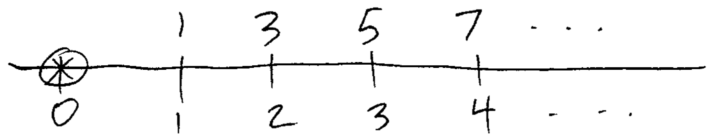
25 This is a typical exam question for increasing perpetuities. Once you have the formula for the perpetuity, you can use the annuity values to solve for the interest rate. Then you can calculate the present value of the perpetuity.

The first step is to write down the series of payments on a time line diagram:



If you know the general formula for a perpetuity that increases, the first step is very quick. For a perpetuity that starts at amount P, and increases by amount Q each year, the present value is $\frac{P}{i} + \frac{Q}{i^2}$.

PV of perpetuity $= \frac{1}{i} + \frac{2}{i^2}$

We will derive this result at the end of the solution. Now you need to use the two annuity certain formulas to solve for the rate of interest:

$$\ddot{s}_{\overline{2n}|} = 72 = \ddot{s}_{\overline{n}|} + (1+i)^n \ddot{s}_{\overline{n}|}$$

$$\ddot{a}_{\overline{n}|} = 6 = (1+i)^{-n} \ddot{s}_{\overline{n}|}$$

$$\ddot{s}_{\overline{n}|} = 6(1+i)^n$$

$$\ddot{s}_{\overline{2n}|} = 6(1+i)^n + (1+i)^n (6)(1+i)^n$$

(25) $\ddot{s}_{\overline{2n}|} = 72 = 6(1+i)^n + [6(1+i)^n](1+i)^n$

let $X = (1+i)^n$

$72 = 6X + 6X^2$

$X^2 + X - 12 = 0$

Now you can use the quadratic formula to solve for the value of X:

For $aX^2 + bX + C = 0 \Rightarrow X = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$1X^2 + 1X - 12 = 0 \Rightarrow X = \dfrac{-1 \pm \sqrt{1 - 4(1)(-12)}}{2(1)}$

$= \dfrac{-1 \pm \sqrt{49}}{2}$

$= \dfrac{-1+7}{2}$ or $\dfrac{-1-7}{2}$

$X = 3.0$ or -4.0

Since the interest rate must be positive, $(1+i)^n = X = 3$.

$\ddot{a}_{\overline{n}|i} = \dfrac{1-v^n}{d} = 6 = \dfrac{1-1/3}{d} \Rightarrow d = \dfrac{2/3}{6} = .1111$

$d = 1 - v = .1111 \Rightarrow v = .8889$

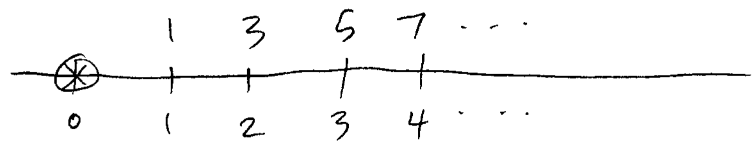
$1+i = 1.125$

PV of Perpetuity $= 1/i + 2/i^2$

$= 1/.125 + 2/(.125)^2 = 8 + 128 = 136$ (B)

(25) There is always an alternate method of solution for problems involving a series of payments. If you do not know the formula for an increasing perpetuity, you can still derive the identical result.

First, write down the payments on a time-line diagram:



Present Value

$$R = v + 3v^2 + 5v^3 + \ldots$$

$$vR = \quad v^2 + 3v^3 + \ldots \qquad \text{(multiply both sides by } v\text{)}$$

$$R(1-v) = v + 2v^2 + 2v^3 + 2v^4 + \ldots \qquad \text{(subtract)}$$

$$vR(1-v) = \quad v^2 + 2v^3 + 2v^4 + \ldots \qquad \text{(multiply both sides by } v\text{)}$$

$$R(1-v)^2 = v + v^2 \qquad \text{(subtract)}$$

Now substitute $d = iv = 1-v$

$$R(iv)^2 = v + v^2$$

$$R = \frac{v}{i^2v^2} + \frac{v^2}{i^2v^2}$$

$$= \frac{1+i}{i^2} + \frac{1}{i^2}$$

$$= \frac{2}{i^2} + \frac{1}{i} \qquad \text{Q.E.D.}$$

In general, you may be better off memorizing the formulas. It takes many steps to derive them, plus careful arithmetic.

26 This problem is unusual. You don't normally see problems on insurance values with a joint and last survivor status.

Start with the usual relationship between insurance and annuity:

$$A_y = 1 - d\ddot{a}_y$$

Now replace the single life y with the joint and last survivor status:

$$A_{\overline{xx}} = 1 - d\ddot{a}_{\overline{xx}}$$
$$= 1 - d(\ddot{a}_x + \ddot{a}_x - \ddot{a}_{xx})$$

Now you can write the annuities as summations, and calculate the values of $\ddot{a}_x$ and $\ddot{a}_{xx}$

$\ddot{a}_x = 1 + vp_x + v^2{}_2p_x + \ldots$
$= 1 + .9/1.05 + (.9/1.05)^2 + \ldots$
$=$ perpetuity due at j
where $1+j = 1.05/.9 = 1.166\overline{6}$
$= \dfrac{1+j}{j} = \dfrac{1.166\overline{6}}{.166\overline{6}}$
$= 7.00$

$\ddot{a}_{xx} = 1 + vp_{xx} + v^2{}_2p_{xx} + \ldots$
$= 1 + (.9)^2/1.05 + (.9)^4/(1.05)^2 + \ldots$
$=$ perpetuity due at k
where $1+k = 1.05/(.9)^2 = 1.296$
$= \dfrac{1+k}{k} = \dfrac{1.296}{.296}$
$= 4.375$

$$\therefore A_{\overline{xx}} = 1 - \frac{.05}{1.05}(7.0 + 7.0 - 4.375)$$
$$= .5417$$

B

2006

27 This is a typical exam question on the joint and last survivor statuses. It is a fairly messy problem, and you must be very careful to avoid arithmetic errors. Since the two payments are actuarially equivalent, you know that the present value of the second payment form is 100,000:

$$100{,}000 = X\ [\, a_{\overline{10}|.04} + v^{10}\,{}_{10}p_{40}(1-{}_{10}p_{40})a_{50:\overline{20}|} \quad \text{1st one alive}$$
$$+ v^{10}(1-{}_{10}p_{40})({}_{10}p_{40})a_{50:\overline{20}|} \quad \text{2nd one alive}$$
$$+ v^{10}({}_{10}p_{40})({}_{10}p_{40})a_{\overline{50:50}:\overline{20}|} \quad \text{both alive}$$

One key point of the problem is that you'll get into trouble if you try to write something like this:

$$100{,}000 = X[\, a_{\overline{10}|.04} + v^{10}\,{}_{10}p_{\overline{40:40}}\ (\ldots)] \quad \text{BAD-WRONG}$$

This won't work, because the annuity values will be different depending on who lives and who dies. Now back to the fun part, which is simplifying the original expression, and expressing it in terms of the factors given in the problem.

$$100{,}000 = X[\, a_{\overline{10}|.04} + 2v^{10}\,{}_{10}p_{40}\,a_{50:\overline{20}|} - 2v^{10}({}_{10}p_{40})^2 a_{50:\overline{20}|}$$
$$+ v^{10}({}_{10}p_{40})^2 (a_{50:\overline{20}|} + a_{50:\overline{20}|} - a_{50:50:\overline{20}|})$$

To expand the joint and last survivor annuity, look at the simplest case:

$$a_{\overline{xy}} = a_x + a_y - a_{xy}$$

(27) When you add on the "n year certain" term, you are chopping off all payments after n years.

$a_{\overline{xy}:\overline{n|}}$

Logically, it should make sense that you can add the "n year certain" term to each of the annuities shown at the bottom of the prior page

$$a_{\overline{xy}:\overline{n|}} = a_{x:\overline{n|}} + a_{y:\overline{n|}} - a_{xy:\overline{n|}}$$

$$100{,}000 = X\Big[a_{\overline{10|}.04} + 2v^{10}\,{}_{10}p_{40}\,a_{50:\overline{20|}} - 2v^{10}({}_{10}p_{40})^2 a_{50:\overline{20|}} + 2v^{10}({}_{10}p_{40})^2 a_{50:\overline{20|}} - v^{10}({}_{10}p_{40})^2 a_{50:50:\overline{20|}}$$

$$= X\Big[a_{\overline{10|}.04} + 2v^{10}\,{}_{10}p_{40}\,a_{50:\overline{20|}} - v^{10}({}_{10}p_{40})^2 a_{50:50:\overline{20|}}\Big]$$

This problem shows that you can save some time doing arithmetic if you try to simplify all the algebra first. Wait until later to start plugging in the numerical values for the factors.

$$100{,}000 = X\Big[a_{\overline{10|}.04} + v^{10}\,{}_{10}p_{40}\big[2a_{50:\overline{20|}} - {}_{10}p_{40}(a_{50:50:\overline{20|}})\big]\Big]$$

$$= X\Big[a_{\overline{10|}.04} + v^{10}\,{}_{10}p_{40}\Big\{2(a_{50} - v^{20}\,{}_{20}p_{50}\,a_{70}) - {}_{10}p_{40}(a_{50:50} - v^{20}({}_{20}p_{50})^2 a_{70:70})\Big\}$$

$$\frac{100{,}000}{X} = a_{\overline{10|}.04} + v^{10}({}_{10}p_{40})\Big\{2(a_{50} - v^{20}\,{}_{20}p_{50}\,a_{70}) - {}_{10}p_{40}(a_{50:50} - v^{20}({}_{20}p_{50})^2 a_{70:70}\Big\}$$

$$= 8.1109 + (1.04)^{-10}(.8848)\Big\{2(12.522 - (1.04)^{-20}(.5217)(6.293)) - .8848(9.695 - (1.04)^{-20}(.5217)^2(4.054)\Big\}$$

$$= 8.1109 + .6756(.8848)\Big\{2(12.522 - 1.498) - .8848(9.695 - .504)\Big\}$$

$$X = (100{,}000 / 16.428) = 6{,}087$$

B

28 This is one of several recent exam questions that test on the assumption of uniform distribution of decrements in the single decrement tables. All of the "old classic" multiple decrement problems prior to 2001 were based on the assumption of uniform distribution of decrements in the multiple decrement tables.

The key to this problem is knowing the integration formulas from the Bowers book, and how to simplify the results:

$$q_x^{(K)} = \int_0^1 {}_tp_x^{(T)} \mu_{x+t}^{(K)}\,dt$$

(simpler problems use 2 decrements)

3 decrements

$$q_x^{(2)} = \int_0^1 {}_tp_x^{(T)} \mu_{x+t}^{(2)}\,dt$$

$$= \int_0^1 {}_tp_x'^{(1)}\,{}_tp_x'^{(3)}\,{}_tp_x'^{(2)}\,\mu_{x+t}^{(2)}\,dt$$

2 decrements

$$q_x^{(2)} = \int_0^1 {}_tp_x^{(T)} \mu_{x+t}^{(2)}\,dt$$

$$= \int_0^1 {}_tp_x'^{(1)}\,{}_tp_x'^{(2)}\,\mu_{x+t}^{(2)}\,dt$$

The key to working this problem is that the value of ${}_tp_x'^{(2)}\mu_{x+t}^{(2)}$ is constant under the assumption of U.D.D, and it equals $q_x'^{(2)}$:

$$q_x^{(2)} = \int_0^1 {}_tp_x'^{(1)}\,{}_tp_x'^{(3)}\,q_x'^{(2)}\,dt$$

$$= q_x'^{(2)} \int_0^1 (1-t\,q_x'^{(1)})(1-t\,q_x'^{(3)})\,dt$$

$$q_x^{(2)} = \int_0^1 {}_tp_x'^{(1)}\,q_x'^{(2)}\,dt$$

$$= q_x'^{(2)} \int_0^1 (1-t\,q_x'^{(1)})\,dt$$

(28) $q_x^{(2)} = q_x^{\prime(2)} \int_0^1 (1 - q_x^{\prime(1)})(1 - {}_tq_x^{\prime(3)})\,dt$

The problem asks for $q_{25}^{(2)}$, so you need to calculate the value of $q_{25}^{\prime(2)}$:

$$q_{25}^{\prime(2)} = 1 - p_{25}^{\prime(2)} = 1 - l_{26}^{\prime(2)}/l_{25}^{\prime(2)} = 1 - 80/100 = .20$$

Based on the assumption of U.D.D. in the single decrement tables, you can say that ${}_tq_x^{\prime(k)} = t(q_x^{\prime(k)})$:

$$q_{25}^{(2)} = q_{25}^{\prime(2)} \int_0^1 [1 - t(q_{25}^{\prime(1)})][1 - t(q_{25}^{\prime(3)})]\,dt$$

You need to calculate the values of $q_{25}^{\prime(1)}$ and $q_{25}^{\prime(3)}$:

$$q_{25}^{\prime(1)} = 1 - \frac{l_{26}^{\prime(1)}}{l_{25}^{\prime(1)}} = 1 - \frac{90}{100} = .10 \qquad q_{25}^{\prime(3)} = 1 - \frac{l_{26}^{\prime(3)}}{l_{25}^{\prime(3)}} = 1 - \frac{70}{100} = .30$$

$$q_{25}^{(2)} = .2 \int_0^1 [1 - .10t][1 - .30t]\,dt$$

$$= .2 \int_0^1 1 - .4t + .03t^2\,dt$$

$$= .2\ [t - .2t^2 + .01t^3]_0^1$$

$$= .2\ (1 - .2(1) + .01(1) - 0)$$

$$= .2(.81)$$

$$= .162$$

C

(28) General formula for solution:

Instead of plugging in the $q_x'^{(k)}$ values, you could leave that step until the end. I'll demonstrate the resulting formulas for both the 2 decrement case and the 3 decrement case:

3 decrements

$$q_x^{(2)} = q_x'^{(2)} \int_0^1 [1-t(q_x'^{(1)})][1-t(q_x'^{(3)})]\,dt$$

$$q_x^{(2)} = q_x'^{(2)} \int_0^1 \left(1-t(q_x'^{(1)}+q_x'^{(3)}) + t^2 q_x'^{(1)} q_x'^{(3)}\right)dt$$

$$= q_x'^{(2)} \left[t - \frac{t^2}{2}[q_x'^{(1)}+q_x'^{(3)}] + \frac{t^3}{3}(q_x'^{(1)})q_x'^{(3)} \right|_0^1$$

2 decrements

$$q_x^{(2)} = q_x'^{(2)} \int_0^1 [1-t(q_x'^{(1)})]\,dt$$

$$q_x^{(2)} = q_x'^{(2)} \int_0^1 (1-t q_x'^{(1)})\,dt$$

$$q_x^{(2)} = q_x'^{(2)} \left[t - \frac{t^2}{2} q_x'^{(1)} \right|_0^1$$

The formulas are usually written in terms of $q_x'^{(k)}$:

$$q_x'^{(2)} = \frac{q_x^{(2)}}{1 - \frac{1}{2}(q_x'^{(1)}+q_x'^{(3)}) + \frac{1}{3}(q_x'^{(1)})(q_x'^{(3)})}$$

$$q_x'^{(2)} = \frac{q_x^{(2)}}{1 - \frac{1}{2} q_x'^{(1)}}$$

These formulas match those shown in the Bowers book for the assumption of U.D.D. in the single decrement tables. There is an exercise that asks for the derivation of the 3 decrement case. Most exam problems test the 2 decrement case.

(28) It is possible to get in the right answer range using a different formula. This formula is valid for either a constant force of mortality, or for U.D.D. in the multiple decrement tables:

$${}_tp_x'^{(k)} = \left[{}_tp_x^{(T)}\right]^{\frac{q_x^{(k)}}{q_x^{(T)}}}$$

Even though this formula is incorrect for the assumption of U.D.D. in the single decrement table, it produces a result of .1615, which is also in answer range C.

If you used this on the exam, you were very lucky it did end up in the correct range. I would not expect that to happen!

29 This is an interesting exam question on various mortality assumptions. You are told that p_x is constant above age 50. This is identical to the assumption of a constant force of mortality:

$${}_tp_x = e^{-\int_x^{x+t} \mu_y \, dy} \quad \text{in general}$$

$${}_tp_x = e^{-t\mu} \quad \text{if constant force of mortality}$$

If $p_x = .75$ for $x \geq 50$, then ${}_tp_x = (.75)^t$.

Below age 50, "deaths are uniformly distributed". That means you have the same number of deaths at each age. This is de Moivre's law, where l_x is a straight line formula.

First you must calculate the value of l_{50}. Then you can determine the number of deaths at each age. Next, you can calculate the value of $l_{45.50}$, and also the value of $l_{55.75}$. Finally, you can calculate the number of deaths between ages 45.50 and 55.75.

$$l_{52} = 270/.75$$
$$l_{51} = 270/(.75)^2$$
$$l_{50} = 270/(.75)^3 = 640$$

2006

(29)

$$l_0 = 1140$$
$$l_{50} = 640$$
$$l_0 - l_{50} = 500$$

$\therefore$ deaths at each age $= 500/50 = 10$

$$l_{49} = l_{50} + d_{49} = 640 + 10$$
$$l_{48} = 640 + 2(10)$$
$$l_{45} = 640 + 5(10)$$
$$= 690$$
$$l_{45.5} = 690 - \tfrac{1}{2}(10)$$
$$= 685$$

$$l_{54} = .75(l_{53})$$
$$l_{55} = (.75)^2(l_{53})$$
$$= (.75)^2(270)$$
$$= 151.88$$
$$l_{55.75} = (.75)^{.75}(151.88)$$ Based on constant $p_x = .75$
$$= 122.40$$

$$l_{45.5} - l_{55.75} = 685 - 122.40$$
$$= 562.60$$

D

30 This is a typical exam question on insurance and death benefits. The key to this problem is writing the formula for the return of premium death benefit correctly. Another key is knowing the definition

$_nE_x = v^n \, _np_x$

Let P be the single premium for this benefit.

$$P = 10{,}000 A_{40} + .5P\left[1.05 v q_{40} + (1.05)^2 v^2 p_{40} q_{41} + \dots + (1.05)^{10} v^{10} \, _9p_{40} q_{49}\right]$$
$$= 10{,}000 A_{40} + .5P\left[q_{40} + {}_1p_{40} q_{41} + \dots + {}_9p_{40} q_{49}\right]$$
$$= 10{,}000 A_{40} + .5P\left[1 - {}_{10}p_{40}\right]$$
$$10{,}000 A_{40} = .5P\left[{}_{10}p_{40} + 1\right]$$

Now you can use $_5E_{40}$ and $_5E_{45}$ to derive the value for $_{10}p_{40}$:

$$_5E_{40} = v^5 \, _5p_{40}$$
$$_5E_{45} = v^5 \, _5p_{45}$$
$$_5E_{40}(_5E_{45}) = v^{10} \, _{10}p_{40}$$

$$_{10}p_{40} = (1.05)^{10}(_5E_{40})(_5E_{45})$$
$$= (1.6289)(.71823)(.71230)$$
$$= .83334$$

$$P = \frac{10{,}000 A_{40}}{.5(_{10}p_{40} + 1)}$$
$$= \frac{10{,}000(.31549)}{.5(1.83334)}$$
$$= 3{,}442$$

B