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SPRING 2008 EA-1
EXAM SOLUTIONS

Revision History:
03/13/13 Revised solutions for problems 5 and 8
02/01/10 Revised solution for problem 15

2008 EA-1 Solutions

1 Even though this looks like a bond problem, it is better to work this as the present value of a geometrically increasing annuity. The first step is to write the series of payments on a time line:

	75	$75(1.03)$	$75(1.03)^2$	$\dots$	$75(1.03)^{18}$	$75(1.03)^{19}$ 1050
0	1	2	3	$\dots$	19	20

The face amount of the bond isn't really used, since the redemption value is what you'll receive after 20 years.

Now, write down the purchase price, which is the present value of the coupons and the redemption amount:

$$PV = 1050(1.0825)^{-20} + 75\left[\frac{1}{1.0825} + \frac{1.03}{(1.0825)^2} + \dots + \frac{(1.03)^{19}}{(1.0825)^{20}}\right]$$

$$= 1050(.20485) + \frac{75}{1.0825}\left[1 + \frac{1.03}{1.0825} + \dots + \left(\frac{1.03}{1.0825}\right)^{19}\right]$$

$$= 215.10 + 69.28\, \ddot{a}_{\overline{20}|j} \quad \text{where } 1+j = 1.0825/1.03 = 1.0510$$

$$= 215.10 + 69.28(1.0510)\, a_{\overline{20}|5.10\%}$$

$$= 215.10 + 69.28(1.0510)(12.3603)$$

$$= 1,115.11$$

D

For the EA-1 exam, I set my HP-12C calculator to always determine the value of immediate annuities. That is why I re-wrote the value of $\ddot{a}_{\overline{20}|j}$ as $(1+j)(a_{\overline{20}|j})$.

2 The key point of this problem is whether you know what a spot rate refers to. It is the yield rate for a zero coupon bond. Using exam condition 10, the interest rate given in the problem should be interpreted as the yield rate for the purchaser of the bond:

$$\text{Bond Price} = 1{,}000\,(1.10)^{-1} + 1{,}000\,(1.10)^{-2}$$

Using the yield curve approach, the cash flow at time 1 would use the one-year spot rate. The cash flow at time two would use the two-year spot rate:

$$\text{Bond Price} = 1{,}000\,(1+x)^{-1} + 1{,}000\,(1.08)^{-2}$$

Now you can solve for the value of x:

$$1000\left[(1.10)^{-1}+(1.10)^{-2}\right] = 1000\left[(1+x)^{-1}+(1.08)^{-2}\right]$$

$$(1.10)^{-1}+(1.10)^{-2}-(1.08)^{-2} = (1+x)^{-1}$$

$$1.7355 - .8573 = (1+x)^{-1}$$

$$1.1387 = 1+x$$

$$13.87\% = x$$

D

3 This is a rare problem on commutation functions, and potentially tricky one at that. The key is whether you know the definition of S_x

$$D_x = v^x l_x \qquad N_x = \sum_{t=0}^{\omega-x} D_{x+t} \qquad S_x = \sum_{t=0}^{\omega-x} N_{x+t}$$

S_x represents a series of increasing payments, starting at age x. S_x/D_x represents an increasing annuity:

Payments: 1 2 3 4 …

Time: 0 1 2 3 …

$(S_x - S_{x+3})/D_x$ represents this series of payments:

Payments: 1 2 3 3 3 … 3 3 3 3

Time: 0 1 2 3 4 … 5 6 7 8

$(S_{x+5} - S_{x+8})/D_x$ represents this series of payments:

Payments: 0 0 0 0 0 1 2 3 3 3 …

Time: 0 1 2 3 4 5 6 7 8 9 …

The difference of those two sets of payments is the answer to the question:

Payments: 1 2 3 3 3 2 1 0 0 …

Time: 0 1 2 3 4 5 6 7 8 …

The total payments equal 15. (D)

4 This is one of the most complicated problems on the EA-1 exam involving multiple life contingencies. Just for good measure, you also have to figure out what the question is asking for: "the present value of Smith's share".

Based on the description in the problem, you can identify the payments to each person, which vary depending on who is alive:

	Age 25 Smith	Age 30 Jones	Age 35 Brown
All 3 alive	2,500	2,500	5,000
Only 2 alive			
Smith + Jones	5,000	5,000	
Smith + Brown	3,750		6,250
Jones + Brown		3,750	6,250
Only 1 alive			
Smith	10,000		
Jones		10,000	
Brown			10,000

I assume that the present value of Smith's share is the present value of their future payments:

$$X = 2500\, a_{25:30:35} + 5{,}000(a_{25:30} - a_{25:30:35}) + 3750(a_{25:35} - a_{25:30:35})$$
$$+ 10{,}000(a_{25} - a_{25:\overline{30:35}})$$

(4) The key to solving the problem is knowing how to use reversionary annuities to write down the formula for X. The last reversionary annuity includes a joint and last survivor annuity, which needs to be simplified:

$$a_{25:\overline{30:35}} = a_{25:30} + a_{25:35} - a_{25:30:35}$$

$$\begin{aligned} X = {} & 2{,}500\, a_{25:30:35} \\ & - 5{,}000\, a_{25:30:35} + 5{,}000\, a_{25:30} \\ & - 3{,}750\, a_{25:30:35} \qquad\qquad + 3{,}750\, a_{25:35} \\ & + 10{,}000\, a_{25:30:35} - 10{,}000\, a_{25:30} - 10{,}000\, a_{25:35} + 10{,}000\, a_{25} \end{aligned}$$

$$\begin{aligned} & = 3{,}750\, a_{25:30:35} - 5{,}000\, a_{25:30} - 6{,}250\, a_{25:35} + 10{,}000\, a_{25} \\ & = 3{,}750(12.68) - 5{,}000(14.92) - 6{,}250(14.48) + 10{,}000(17.95) \\ & = 61{,}950 \end{aligned}$$

B

5 This is a fairly typical question on accumulating fund balances. It doesn't really have much to do with the force of interest, other than the basic identity:

$e^{\delta} = 1+i \qquad e^{\delta/2} = (1+i)^{\frac{1}{2}}$

Based on the initial investment, you have this formula for the fund value after 2n years:

$(3000(1+i)^n - 2{,}000)(1+i)^n = 225{,}000$

This can be solved using the quadratic equation. Rewrite the original formula as follows, based on $K = (1+i)^n$

$(3000K - 2{,}000)K = 225{,}000$

$3000K^2 - 2000K - 225{,}000 = 0$

Using the quadratic formula, you have

$aK^2 + bK + C = 0$

$$K = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2{,}000 \pm \sqrt{(2{,}000)^2 - 4(3000)(-225{,}000)}}{2(3{,}000)}$$

$$= \frac{2{,}000 \pm \sqrt{4{,}000{,}000 + 2{,}700{,}000{,}000}}{6{,}000}$$

$$= \frac{2{,}000 \pm 52{,}000}{6{,}000}$$

The negative value is spurious, and should be ignored

$K = (2{,}000 + 52{,}000)/6{,}000 = 9.0$

(5) $(1+i)^n = 9.0$

$(1+i)^{n/2} = 3.0$

Now you can calculate the value of X:

$$X = (3000(1+i)^{n/2} - 2{,}000)(1+i)^{n/2}$$
$$= (3000\,(3.0) - 2{,}000)(3.0)$$
$$= 7{,}000(3)$$
$$= 21{,}000$$

A

6 This is a typical problem involving identities for interest only annuities. You can solve for the interest rate directly:

$$s_{\overline{n}|} = 10 \qquad \ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|} = 11$$

$$(1+i)(10) = 11$$

Now you can approach the problem two ways. One is to solve for the value of n, but that is a "brute force" approach. The other is to express the $n|\ddot{a}_{\overline{2n}|}$ in terms of the items already given:

$$n|\ddot{a}_{\overline{2n}|} = v^n(\ddot{a}_{\overline{2n}|}) = v^n(\ddot{a}_{\overline{n}|} + v^n\ddot{a}_{\overline{n}|})$$
$$= v^n(v^n\ddot{s}_{\overline{n}|} + v^{2n}\ddot{s}_{\overline{n}|})$$
$$= v^{2n}(\ddot{s}_{\overline{n}|}) + v^{3n}(\ddot{s}_{\overline{n}|})$$
$$= 11(v^{2n} + v^{3n})$$

Now you can solve for the value of $(1+i)^n$ based on the original identity:

$$s_{\overline{n}|i} = 10 = \frac{(1+i)^n - 1}{i} = \frac{(1+i)^n - 1}{.10}$$

$$1 = (1+i)^n - 1 \Rightarrow (1+i)^n = 2 \Rightarrow v^n = 1/2$$

$$n|\ddot{a}_{\overline{2n}|} = 11[(1/2)^2 + (1/2)^3]$$
$$= 11[.25 + .125]$$
$$= 4.125$$

C

7 This is a typical exam question involving identities for interest only annuities. Based on the initial formula, you can derive the interest rate.

$\ddot{a}_{\overline{10}|} - \ddot{a}_{\overline{9}|} = .50$

$\ddot{a}_{\overline{10}|} = 1 + a_{\overline{9}|}$

$\ddot{a}_{\overline{9}|} = (1+i)a_{\overline{9}|}$

$(1+a_{\overline{9}|}) - (1+i)a_{\overline{9}|} = .50 = 1 - i a_{\overline{9}|}$

$= 1 - (1 - v^9)$

$\therefore v^9 = .50 \Rightarrow (1+i)^9 = 2 \quad \Rightarrow \quad i = 8.006\%$

$$X = \frac{(s_{\overline{19}|} - s_{\overline{18}|})^2}{(\ddot{s}_{\overline{9}|} - \ddot{s}_{\overline{8}|})^3} = \frac{(41.472 - 37.472)^2}{(1.08006)^3 (s_{\overline{9}|} - s_{\overline{8}|})^3} = \frac{16}{(1.260)(6.350)}$$

$= 2.0$

D

Another approach takes a little bit longer, but requires fewer calculations. You can simplify the accumulated annuity formulas:

$s_{\overline{19}|} - s_{\overline{18}|} \Rightarrow s_{\overline{19}|} - 1.0 = \ddot{s}_{\overline{18}|}$

$s_{\overline{19}|} - s_{\overline{18}|} = 1 + \ddot{s}_{\overline{18}|} - s_{\overline{18}|} = 1 - s_{\overline{18}|} + (1+i)s_{\overline{18}|}$

$= 1 + i(s_{\overline{18}|}) \qquad = (1+i)^{18}$

$= (v^9)^{-2} \qquad = 4.0 \qquad (s_{\overline{19}|} - s_{\overline{18}|})^2 = 16.0$

$\ddot{s}_{\overline{9}|} - \ddot{s}_{\overline{8}|} = \ddot{s}_{\overline{9}|} - (s_{\overline{9}|} - 1.0)$

$= (1+i)s_{\overline{9}|} - s_{\overline{9}|} + 1.0$

$= 1 + i(s_{\overline{9}|}) \qquad = (1+i)^9$

$= (v^9)^{-1} \qquad = 2.0 \qquad (\ddot{s}_{\overline{9}|} - \ddot{s}_{\overline{8}|})^3 = 8.0$

8 The key to this problem is knowing two things:
(1) what is the series of payments made for a serial bond?
(2) what is the formula to calculate the duration?

The serial bond is a set of four bonds, with a total face amount of 1,000. Each bond has a face amount of 250 with coupons of 12.50 = 5% (250). The bonds are redeemed at four different dates. You should write all the payments on a time line diagram

BOND #				
1	12.50 + 250			
2	12.50	12.50 + 250		
3	12.50	12.50	12.50 + 250	
4	12.50	12.50	12.50	12.50 + 250
0	1	2	3	4

The regular duration ($\bar{d}$) for a series of payments represents the weighted average future years to each payment (R_t), where the weight is the present value of the payment

$$\bar{d} = \Sigma(t v^t R_t) / \Sigma(v^t R_t)$$ NOT formula for modified duration!

$$= \frac{1 \cdot v\,(262.50 + 37.50) + 2 \cdot v^2 (262.50 + 25.00) + 3 \cdot v^3 (262.50 + 12.50) + 4 v^4 (262.50)}{262.50\, a_{\overline{4}|} + 12.50\,(a_{\overline{1}|} + a_{\overline{2}|} + a_{\overline{3}|})}$$

The face value of the bonds cancels out of both the numerator and denominator, which simplifies the calculations.

(8) modified duration $= \bar{d}/(1+i) = \bar{d}/(1.04)$
(not asked in this exam question)

Regular duration

$$\bar{d} = \frac{1\cdot v\,[F+4(.05)F]+2v^2[F+3(.05)F]+3v^3[F+2(.05)F]+4v^4[F+.05F]}{F a_{\overline{4}|} + .05F\,(a_{\overline{1}|}+a_{\overline{2}|}+a_{\overline{3}|}+a_{\overline{4}|})}$$

$$= \frac{F(Ia_{\overline{4}|}) + .05F\,(4v+6v^2+6v^3+4v^4)}{F(a_{\overline{4}|}) + .05F\,(Da_{\overline{4}|})}$$

To evaluate the duration, you must know the formulas for increasing and decreasing annuities

$$Ia_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} = \frac{(1+i)a_{\overline{n}|} - nv^n}{i}$$

$$Da_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$$

$$\bar{d} = \frac{\dfrac{(1.04)a_{\overline{4}|} - 4v^4}{.04} + .05\,(4v+6v^2+6v^3+4v^4)}{a_{\overline{4}|.04} + .05\left(\dfrac{4-a_{\overline{4}|}}{.04}\right)}$$

$$= \frac{\dfrac{1.04(3.630)-3.4192}{.04} + .05(3.8462+5.5473+5.3340+3.4192)}{3.630+.05(9.2526)}$$

$$= 9.8042/4.0925$$

$$= 2.396$$

C

9 This looks like a bond problem, but it is more like a sinking fund problem. One confusing aspect is that you are told the yield rate is 5%, but you are also solving for the overall rate of return.

The key idea is that you use the yield rate to determine the initial price of the bond. Then you accumulate the bond coupons and the redemption value at the 10% per year rate. Then you can solve for the overall rate of return, since that will accumulate the original bond purchase price to the same value.

First you need to convert the semi-annual rates, since the coupons are paid semi-annually:

$(1+k)^2 = 1.10 \Rightarrow 1+k = 1.0488 \quad k = 4.88\%$

$(1+j)^2 = 1.05 \Rightarrow 1+j = 1.0247 \quad j = 2.47\%$

Coupons = 4.0%(10,000) per annum, paid semi-annually

Bond price $= 200\, a_{\overline{40}|2.47\%} + 10{,}500(1.0247)^{-40}$

$= 5{,}046.44 + 3{,}957.34 = 9003.77$

Accumulated

Value of Coupons $= 200\, s_{\overline{40}|4.88\%} = 23{,}469.11$

AV + redemption $= 33{,}969.11 = 23{,}469.11 + 10{,}500$

Now you can solve for the total rate of return (annual rate)

$9003.77(1+R)^{20} = 33{,}969.11$

$(1+R)^{20} = 3.7728 \Rightarrow R = 6.86\%$ (B)

10 This problem is unusual, since you are not given any actuarial factor values for the calculation. You should write formulas for each of the premiums described – this should simplify to a basic algebra question!

$$A_{x:\overline{25|}} = 272.70\,\ddot{a}_{x:\overline{5|}} + 746.00\,(v^5\,{}_5p_x)(\ddot{a}_{x+5:\overline{20|}})$$

$$= 136.10\,\ddot{a}_{x:\overline{5|}} + 846.00\,(v^5\,{}_5p_x)(\ddot{a}_{x+5:\overline{20|}})$$

$$= P\,\ddot{a}_{x:\overline{25|}}$$

In order to determine the value of P, you should express the premiums solely in terms of $\ddot{a}_{x:\overline{5|}}$ and $\ddot{a}_{x:\overline{25|}}$:

$$A_{x:\overline{25|}} = 272.70\,\ddot{a}_{x:\overline{5|}} + 746.00(\ddot{a}_{x:\overline{25|}} - \ddot{a}_{x:\overline{5|}})$$

$$= 136.10\,\ddot{a}_{x:\overline{5|}} + 846.00(\ddot{a}_{x:\overline{25|}} - \ddot{a}_{x:\overline{5|}})$$

$$= P\,\ddot{a}_{x:\overline{25|}}$$

$$P\,\ddot{a}_{x:\overline{25|}} = 272.70\,\ddot{a}_{x:\overline{5|}} + 746.00\,(\ddot{a}_{x:\overline{25|}} - \ddot{a}_{x:\overline{5|}})$$

$$P\,\ddot{a}_{x:\overline{25|}} = 136.10\,\ddot{a}_{x:\overline{5|}} + 846.00\,(\ddot{a}_{x:\overline{25|}} - \ddot{a}_{x:\overline{5|}})$$

At ~~this~~ point, you have two equations in three unknowns. You can reduce this to two equations in two unknowns by dividing both sides by $\ddot{a}_{x:\overline{25|}}$. Simply treat the ratio of the annuities as a new unknown:

$$P = 272.70(R) + 746.00(1-R)$$

$$P = 136.10\,(R) + 846.00(1-R)$$

(10) Since you are solving for P, the first step is calculating the value of R. If you subtract the two equations, you get a formula for R only:

$$0 = 136.60R - 100(1-R)$$
$$100 = 236.60R$$
$$.4227 = R$$

Substitute this into the prior formula to solve for P:

$$P = 272.70(.4227) + 746.00(.5773)$$
$$= 115.26 + 430.70$$
$$= 545.96$$

D

11 The present value of the annuity and death benefit correspond to this series of payments:

Death Ben		5000(19)	5000(18)	. . .	5000(2)	5000(1)	
Annuity	5000	5000	5000	. . .	5000	5000	
	40	41	42	. . .	58	59	60

The commutation function R_x corresponds to an increasing insurance. But the series of payments for this problem is a decreasing insurance. That can be represented as a level insurance minus an increasing insurance:

	5000(20) −5000(1)	5000(20) −5000(2)		5000(20) −5000(18)	5000(20) −5000(19)	5000(20) −5000(20)
40	41	42	. . .	58	59	60

$$Y = 5000\,\ddot{a}_{40:\overline{20}|} + 5000\left[20\,A^{1}_{40:\overline{20}|} - IA^{1}_{40:\overline{20}|}\right]$$

Since you only have commutation functions at age 60, you need the two offsetting insurance payments at age 60.

There are two commutation functions that you are not given: D_x and M_x. You can derive the values you need at ages 40 and 60:

$D_{40} = N_{40} - N_{41} = 1{,}810$ $\quad M_{40} = R_{40} - R_{41} = 752$ $\quad M_{60} = R_{60} - R_{61} = 420$

(11) $$Y = 5000\left[\frac{N_{40}-N_{60}}{D_{40}} + 20\left(\frac{M_{40}-M_{60}}{D_{40}}\right) - \left(\frac{R_{40}-R_{60}-20M_{60}}{D_{40}}\right)\right]$$

$$= 5000\left[\frac{31{,}309-7147+20(752-420)-(17{,}169-5105-20(420))}{1810}\right]$$

$$= \frac{5000}{1810}\left[24{,}162+6{,}640-3{,}664\right]$$

$$= 74{,}967$$

D

12 The problem asks for you to calculate the complete expectation of life for 10 people, all at age 90. Under UDD, the formula for $\mathring{e}_x$ is

$$\mathring{e}_x = e_x + .50 \text{ under UDD}$$

$$e_x = p_x + {}_2p_x + {}_3p_x + {}_4p_x + \ldots$$

$$= \frac{l_{x+1} + l_{x+2} + l_{x+3} + \cdots}{l_x}$$

$$= \frac{150 + 120 + 90 + 70 + 50 + 35 + 20 + 10 + 5}{210}$$

$$= \frac{550}{210}$$

$$= 2.62$$

$$\mathring{e}_x = 3.12$$

$$10\mathring{e}_x = 31.2$$

B

13 This is a typical exam question on loans. If you can set up the loan amortization schedule, this is not a difficult problem to work:

Payment	Amount	Interest paid	Principal Paid	Remaining Loan
				$100a_{\overline{60}\rceil}$
1	100	$100(1-v^{60})$	$100v^{60}$	$100a_{\overline{59}\rceil}$
2	100	$100(1-v^{59})$	$100v^{59}$	$100a_{\overline{58}\rceil}$
⋮	⋮	⋮	⋮	
20	100	$100(1-v^{41})$	$\underline{100v^{41}}$	$100a_{\overline{40}\rceil}$

$$\Sigma = 100(a_{\overline{60}\rceil}-a_{\overline{40}\rceil}) = 1461$$

The principal paid in the second 20 months is $100v^{40}+\ldots+100v^{21} = 100(a_{\overline{40}\rceil}-a_{\overline{20}\rceil}) = 1655.$

$$a_{\overline{60}\rceil}-a_{\overline{40}\rceil} = 14.61 \qquad\qquad a_{\overline{40}\rceil}-a_{\overline{20}\rceil} = 16.55$$

$$v^{40}-v^{60} = 14.61i \qquad\qquad v^{20}-v^{40} = 16.55i$$

$$i = v^{20}(v^{20}-v^{40})/14.61$$

Now you can substitute the value of $v^{20}-v^{40}$ from the second equation, and solve for the value of v:

$$i = \frac{v^{20}}{14.61}(16.55i)$$

$$v^{20} = .8828$$

$$v = .9938$$

$i = .63\%$ per month

Problem asks for annual rate! $(1+i)^{12} = 1.0777$ D

14 The key point of this question is whether you know the definition of the term "unrealized gain or loss". That is the increase in assets that is reflected in the market value, but not the book value.

That is quite different than the "realized gain or loss". That represents the effect of selling an investment, and it affects both the book value and the market value.

You are told that the cumulative unrealized G/L is negative 10,000 at 12/31/06. That means the book value at that date is 10,000 higher than market value:

12/31/06 BV = 1,000,000 + 10,000 = 1,010,000

You are told that the unrealized G/L during 2007 is 5,000. That means that the cumulative unrealized G/L is negative 5,000 at 12/31/07:

12/31/06 cumulative unrealized G/L + 5,000 = 12/31/07 cumulative unrealized G/L

-10,000 + 5,000 = -5,000

The book value at 12/31/07 is 5,000 higher than the market value:

12/31/07 BV = 1,150,000 + 5,000 = 1,155,000 D

15 You need to calculate the future working lifetime from age 22 to age 62. You are told that there are decrements of 3% at the end of each year. Everyone retires at exact age 62, and there is one extra decrement of 5% at age 42.

This is similar to an expectation of life calculation, but there is a trick. The years of service earned in the first year is 1.0 instead of .97. The reason is that the decrement occurs at the end of the year.

If you ignore the 5% decrement at age 42, you can write an expression for the future working lifetime:

$$e_{22:\overline{40|}} = 1 + .97 + (.97)^2 + \ldots + (.97)^{39} \quad \text{ignoring 5\% decrement}$$
$$= \ddot{a}_{\overline{40|}j} \quad \text{where } 1+j = \frac{1}{.97} = 1.0309$$
$$= (1.0309)\, a_{\overline{40|}3.09\%}$$

If you allow for the 5% decrement at age 42, the portion of the person who exits at that age has only had 20 years of service at exit. The 95% of the person that does not ~~leave~~ at age 42 has 40 years at exit:

$$e_{22:\overline{40|}} = .95(\ddot{a}_{\overline{40|}3.09\%}) + .05(\ddot{a}_{\overline{20|}3.09\%})$$
$$= .95(1.0309)(22.7720) + .05(1.0309)(14.7506)$$
$$= 22.3024 + .7603$$
$$= 23.0628$$

B

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16 This is a typical question involving refinancing. The steps in the solution are

1. Calculate initial monthly payment based on 30 years
2. Determine O/S mortgage after 60 payments
3. Reduce O/S mortgage by extra payment of 50,000
4. Calculate revised payment based on 10 years

Step 1: Initial payment $= \dfrac{200,000}{a_{\overline{360}|.5\%}} = 1,199.10$

Step 2: O/S mortgage after 60 payments $= 1199.10\left(a_{\overline{300}|.5\%}\right)$
$= 186,108.71$

Step 3: Reduced mortgage after payment $= 186,108.71 - 50,000$
$= 136,108.71$

Step 4: Revised payment using 10 years and 5% interest
$\dfrac{136,108.71}{a_{\overline{120}|.4167\%}} = 1,443.64$

B

2008

17 This is a typical exam question on time weighted rate of return. The key idea is that the time weighted return is measured between the points in time where a cash flow occurs. It is based on the products of ratios of market value, where each ratio measures the increase in market value since the prior cash flow.

You need to set up the market values and cash flows on a time line diagram:

MV Before	20,000	24,000	X	29,000	34,000
Cash flow	3,000	3,000	3,000	3,000	
	12-31 2007	03-31 2008	06-30 2008	09-30 2008	12-31 2008
MV After	23,000	27,000	X+3,000	32,000	

The problem gives you the market values before each cash flow, so you need to develop the "after" cash flow market values. Now the time weighted rate of return is calculated using the ratios of the market values:

20,000	24,000	X	29,000	34,000
23,000	27,000	X+3,000	32,000	

$$(1+7.0\%) = \frac{24{,}000}{23{,}000}\left(\frac{X}{27{,}000}\right)\left(\frac{29{,}000}{X+3{,}000}\right)\left(\frac{34{,}000}{32{,}000}\right)$$

(17) $1.07 = \left(\frac{24}{23}\right)\left(\frac{X}{X+3{,}000}\right)\left(\frac{29}{27}\right)\left(\frac{34}{32}\right)$

$= 1.0435\left(\frac{X}{X+3{,}000}\right)(1.0741)(1.0625)$

$1.07(X+3{,}000) = 1.1908X$

$X + 3{,}000 = 1.1129X$

$3{,}000 = .1129X$

$X = 26{,}568$

(C)

18 This problem is basically a probability calculation. You need to calculate the probability that the person age 55 will withdraw at 55, and also survive to any later age (prior to 60) and then withdraw:

$$P = q_{55}^{(w)} + p_{55}^{(T)}\, q_{56}^{(w)} + {}_2p_{55}^{(T)}\, q_{57}^{(w)} + {}_3p_{55}^{(T)}\, q_{58}^{(w)} + {}_4p_{55}^{(T)}\, q_{59}^{(w)}$$

$$p_{55}^{(T)} = 1 - q_{55}^{(w)} - q_{55}^{(d)} - q_{55}^{(r)} = 1 - .070 - .025 - .150 = .7550$$

$$p_{56}^{(T)} = 1 - .050 - .029 - .100 = .8210$$

$$p_{57}^{(T)} = 1 - .030 - .033 - .100 = .8370$$

$$p_{58}^{(T)} = 1 - .020 - .037 - .200 = .7430$$

$$P = .07 + .7550(.05) + .755(.821)(.03) + .755(.821)(.837)(.02) + .755(.821)(.837)(.743)(.01)$$

$$= .07 + .0378 + .0186 + .0104 + .0039$$

$$= .1406$$

E

19 This is a relatively simple probability question. One simplifying factor is that the older participant must die between 10 and 15 years from today. But the younger participant must survive the first 15 years.

This means there are no cases where both participants would die within the same year.

$$X = ({}_5p_{50} - {}_{15}p_{50})({}_{15}p_{40})(1 - {}_{10}p_{55})$$

You need to use the information given to derive the value of ${}_5p_{50}$ and ${}_{15}p_{40}$

$${}_5p_{50} = \frac{l_{55}}{l_{50}} \qquad {}_{15}p_{50} = .75 = \frac{l_{65}}{l_{50}} \qquad {}_{10}p_{55} = .80 = \frac{l_{65}}{l_{55}}$$

$$= \frac{{}_{15}p_{50}}{{}_{10}p_{55}} = \frac{.75}{.80}$$

$${}_{15}p_{40} = \frac{l_{55}}{l_{40}} \qquad {}_{10}p_{40} = .90 = \frac{l_{50}}{l_{40}}$$

$$= \left(\frac{l_{55}}{l_{50}}\right)\left(\frac{l_{50}}{l_{40}}\right) = \left(\frac{.75}{.80}\right)(.90)$$

$$X = \left(\frac{.75}{.80} - .75\right)\left(\frac{.75}{.80}\right)(.90)(1 - .80)$$

$$= .1875\,(.8438)(.20)$$

$$= .0316$$

(D)

20 This is a very straightforward question on actuarial equivalence. The key point is whether you know the meaning of the annuity symbols given in the problem.

The two annuities are actuarially equivalent, which means they have the same present values. One minor trick to the problem is that annuity 1 includes a 5 year certain period.

$$PV1 = 12(1000)\left(\ddot{a}^{(12)}_{\overline{5}|.05} + {}_5|\ddot{a}^{(12)}_{60}\right)$$
$$PV2 = 12X\left({}_5|\ddot{a}^{(12)}_{60}\right)$$

$$\ddot{a}^{(12)}_{\overline{5}|.05} = \frac{1}{12}\left(\ddot{a}_{\overline{60}|j}\right) \quad \text{where } (1+j)^{12} = 1.05 \Rightarrow j = .4074\%$$
$$= \frac{1}{12}(1.004074)(a_{\overline{60}|.4074\%})$$
$$= 4.4459$$

Note that I use the HP-12C calculator, and I always have it set to calculate an immediate annuity (not DUE).

$$PV1 = PV2$$
$$12X(8.88) = 12(1000)(4.4459 + 8.88)$$
$$X = 1000(4.4459 + 8.88)/8.88$$
$$= 1500.66$$

C

21 The key to this problem is reading it carefully. The q_{60} of .0070 includes the other q_{60}, since it is for death due to ALL causes.

The insurance premium is simply $v q_{60}$. You need to reflect the "double indemnity" clause for death due to accidental causes:

$$
\begin{aligned}
P &= v(.0070 - .0005)(1{,}000{,}000) \\
&\quad + v(.0005)(2)(1{,}000{,}000) \\
&= (1.06)^{-1}(.0070 + .0005)(1{,}000{,}000) \\
&= 7{,}075.47
\end{aligned}
$$

D

22 The key to this problem is understanding the notation for select and ultimate tables. You can write the probability in terms of $l_{[x]}$ values:

$$P = \frac{l_{[50]+4} - l_{58}}{l_{[50]+2}}$$

The life was at age 50 at the beginning of the select period. They are currently age 52, will survive until age 54, and will die by age 58.

$$P = \frac{(100-50-4/2)-(103-50-8)}{(100-50-2/2)}$$

$$= \frac{48-45}{49}$$

$$= .0612$$

C

23 This seems like it should be straightforward, but you have to be careful. Based on uniform distribution of death, there are various formulas you must know:

$${}_tq_x = t(q_x)$$

$${}_{1-t}q_{x+t} = \frac{(1-t)q_x}{1-t(q_x)}$$

The key to this problem is that the formulas above are only valid for values of $t < 1$. The question asks for the value of $q_{x+.5}$, which crosses over from year of age x to year of age x+1. You need to rewrite this based on death during the interval $[x, x+1]$ and $[x+1, x+2]$:

$$q_{x+.5} = {}_1q_{x+.5} = {}_{.5}q_{x+.5} + (1 - {}_{.5}q_{x+.5})({}_{.5}q_{x+1})$$

$${}_{.5}q_{x+.5} = \frac{.5(q_x)}{1-.5(q_x)} = \frac{.5(.40)}{1-.5(.40)} = \frac{.20}{.80} = .25$$

$${}_{.5}q_{x+1} = .5(q_{x+1})$$
$$= .25$$

$$q_{x+.5} = .25 + (1-.25)(.25)$$
$$= .4375$$

C

24 The key to this problem is knowing the meaning of the various symbols given in the problem. After that understanding, you may have to think a while to figure out how to put those values together to produce the value of $\ddot{a}_x$.

One way to approach this is to write the various items in terms of commutation functions:

$$_{10}E_x = \frac{D_{x+10}}{D_x} = .40$$

$$_{10|}a_x = \frac{N_{x+11}}{D_x} = 7.0$$

$$\ddot{s}_{x:\overline{10}|} = \frac{N_x - N_{x+10}}{D_{x+10}} = 15.0$$

$$\ddot{a}_x = \frac{N_x}{D_x}$$

The key idea is that you need to be able to get rid of N_{x+10}. Luckily you have $N_{x+10} = D_{x+10} + N_{x+11}$:

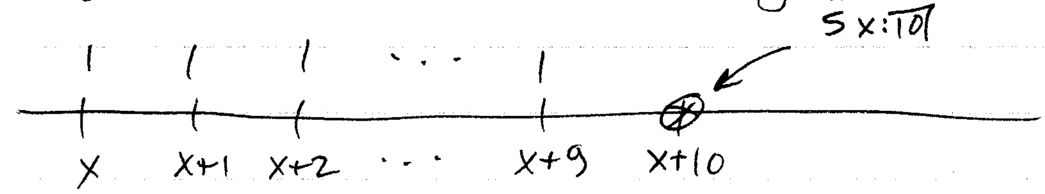
$$\frac{N_{x+10}}{D_x} = .40 + 7.0 = 7.40$$

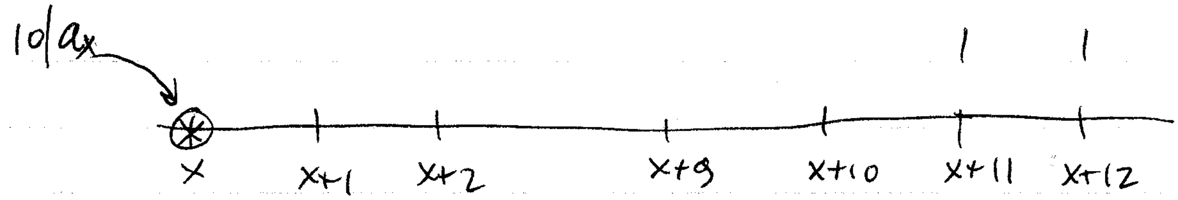
$$\frac{N_x - N_{x+10}}{D_x} = 15.0\left(\frac{D_{x+10}}{D_x}\right) = 6.0$$

$$\ddot{a}_x = N_x / D_x = 7.40 + 6.0 = 13.40$$

E

(24) The alternate method of solution is to write some payments down on a time line diagram, and do it algebraically:





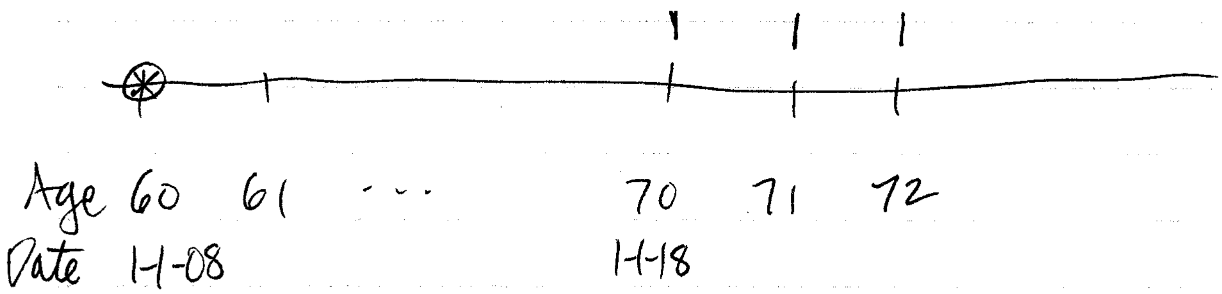
$$_{10|}a_x + a_{x:\overline{10|}} = a_x$$
$$_{10|}a_x + {}_{10}E_x = {}_{9|}a_x$$
$$_{9|}a_x + \ddot{a}_{x:\overline{10|}} = \ddot{a}_x$$

$$\ddot{a}_x = {}_{10|}a_x + {}_{10}E_x + \ddot{a}_{x:\overline{10|}}$$
$$= 7.0 + .40 + \ddot{s}_{x:\overline{10|}}\left(\frac{D_{x+10}}{D_x}\right)$$
$$= 7.0 + .40 + 15.0(.40)$$
$$= 13.40$$

E

2008

25 This is an unusual question where the mortality follows De Moivre's law. The first step is too write down the annuity payments on a time line diagram:



To calculate the given probability, you need to first calculate thee present value of an annuity that starts in 2018. You can't use thee typical formula based on De Moivre's law:

$$a_x = \frac{n - \ddot{a}_{\overline{n}|i}}{n \cdot i}$$

The reason is that the value is indeterminate with an interest rate of zero. You need to calculate the present value based on first principles, and using $l_x = 100 - x$:

$$PV = (l_{70} + l_{71} + l_{72} + \ldots)/l_{60}$$
$$= (30 + 29 + 28 + 27 + \ldots + 2 + 1)/l_{60}$$

As we did in an earlier problem, you can apply the formula for the sum of the first n integers: $\sum_1^n i = \frac{n(n+1)}{2}$

(25) $PV = \dfrac{30(31)/2}{40}$

$= 465/40$

$= 11.625$

The probability Y that the payments exceed 11.625 is the probability of survival to age 81. At that age, they would receive the 12^{th} payment, for ages 70 through 81:

$Y = \dfrac{l_{81}}{l_{60}}$

$= \dfrac{19}{40}$

$= .475$

C

26 This is a simple present value calculation. The key is to write down the formulas for the two present values.

1-1-07 Age 45
1-1-07 Service 10
Projected Benefit $50(12)(65-35) = 18{,}000$ (not necessary)

$$1\text{-}1\text{-}07\ PVB = v^{20}\,{}_{20}p_{45}\,18{,}000\,\ddot{a}^{(12)}_{65}$$
$$1\text{-}1\text{-}08\ PVB = v^{19}\,{}_{19}p_{46}\,18{,}000\,\ddot{a}^{(12)}_{65}$$

$$\frac{1\text{-}1\text{-}08\ PVB}{1\text{-}1\text{-}07\ PVB} = \frac{1+i}{p_{45}} = \frac{v^{19}\,{}_{19}p_{46}}{v^{20}({}_{19}p_{46})\,p_{45}}$$

$$\frac{1.06}{p_{45}} = \frac{28{,}174}{26{,}475}$$

$p_{45} = .9961$

$q_{45} = .0039$ D

As noted above, you don't need to calculate the amount of the projected benefit at age 65.

2008

Problem 27

(Solution not finalized)

Problem 27 - continued

(Solution not finalized)

28 The key to working this problem is understanding how to use the spot rates. These rates are the yield to maturity for a zero coupon bond. You should use the various spot rates to discounts the coupons and the redemption value of the bond.

The typical formula for the price of a bond is
$P = Fr(a_{\overline{n}|i}) + Cv^n$

In this problem, you must split the coupons into three sets, based on the number of years till payment. Then use the given spot rate for discounting

$$X = .05(1000)\left[a_{\overline{5}|.05} + (a_{\overline{15}|.07} - a_{\overline{5}|.07}) + (a_{\overline{20}|.09} - a_{\overline{15}|.09})\right] + 1000(1.09)^{-20}$$

By using the difference in the annuities, you are "carving out" the previous coupons.

$$X = .05(1000)\left[4.3295 + (9.1079 - 4.1002) + (9.1286 - 8.0607)\right] + 178.43$$
$$= 698.68$$

B

29 This is a fairly typical problem on insurance. You need to use the given factors to calculate the death benefit described.

One approach is to use commutation functions for everything. The other is to develop the insurance functions directly. This is actually a better idea!

Start by writing down the death benefits on a time line diagram:

100,000 ⟶ 100,000 200,000 ⟵⟶ 200,000 50,000 ⟶

Age 20   40   60

You are given values for both a term insurance and an endowment insurance. You can use the value of the pure endowment to change the endowment insurance into a term insurance:

$$\begin{aligned} A^{1}_{20:\overline{40|}} &= A_{20:\overline{40|}} - {}_{40}E_{20} \\ &= .1068 - .0901 \\ &= .0167 \end{aligned}$$

Now you can write a formula for the death benefits shown above. The best way is to start with 50,000 for life, and add on 150,000 for the first 40 years. Then subtract 100,000 for the first 20 years.

(29) $P = 50{,}000\, A_{20} + 150{,}000\, A^{1}_{20:\overline{40}|} - 100{,}000\, A^{1}_{20:\overline{20}|}$

$= 50{,}000(.0454) + 150{,}000(.0167) - 100{,}000(.0081)$

$= 2{,}270 + 2{,}505 - 810$

$= 3{,}965$

(D)

The alternative approach is to write the given factors in terms of commutation functions. But there is no way to do anything significantly different in the solution:

$A_{20} = M_{20}/D_{20} = .0454$

$A^{1}_{20:\overline{20}|} = \dfrac{M_{20} - M_{40}}{D_{20}} = .0081$

$A_{20:\overline{40}|} = \dfrac{M_{20} - M_{60} + D_{60}}{D_{20}} = .1068$

${}_{40}E_{20} = \dfrac{D_{60}}{D_{20}} = .0901$

From the diagram on the prior page, you can write down the present value of the death benefits

$P = (100{,}000 M_{20} + 100{,}000 M_{40} - 150{,}000 M_{60}) / D_{20}$

To directly calculate this value, you need to solve for the values of M_{40}/D_{20} and M_{60}/D_{20}:

$A_{20} - A^{1}_{20:\overline{20}|} = M_{40}/D_{20} = .0373$

$A_{20} - A^{1}_{20:\overline{40}|} = M_{60}/D_{20} = A_{20} - (A_{20:\overline{40}|} - {}_{40}E_{20}) = .0287$

$P = 100{,}000(.0454) + 100{,}000(.0373) - 150{,}000(.0287)$

$= 4{,}540 + 3{,}730 - 4{,}305$

$= 3{,}965$

30 This is a typical question on multiple decrement tables. The key idea is being able to interpret the symbol that you are trying to value.

Z represents the probability that someone survives for 2 years, then succumbs to decrement 2 at either age 48 or age 49:

$$Z = {}_2p_{46}^{(T)} \left(q_{48}^{(2)} + p_{48}^{(T)} q_{49}^{(2)} \right)$$

$$p_{46}^{(T)} = 1 - q_{46}^{(1)} - q_{46}^{(2)} = 1 - .0244 - .10 = .8756$$
$$p_{47}^{(T)} = 1 - q_{47}^{(1)} - q_{47}^{(2)} = 1 - .0273 - .09 = .8827$$
$$p_{48}^{(T)} = 1 - q_{48}^{(1)} - q_{48}^{(2)} = 1 - .0309 - .08 = .8891$$

Since you are given probabilities (not rates of decrement) you calculate $p_x^{(T)}$ as $1 - q_x^{(1)} - q_x^{(2)}$.

$$Z = p_{46}^{(T)} p_{47}^{(T)} \left[q_{48}^{(2)} + p_{48}^{(T)} q_{49}^{(2)} \right]$$
$$= .8756(.8827)\left[.08 + .8891(.07)\right]$$
$$= .1099$$

C

31 You need to apply the projection scale to all of the q_x values given to update them to 2008. Then you can calculate the value of Y.

x	$q_x^{[2000]}$	AA_x	$q_x^{[2008]} = q_x^{[2000]}(1-AA_x)^8$
35	.000475	.011	.000435 = .000475(.989)^8
36	.000514	.012	.000467 = .000514(.988)^8
37	.000554	.013	.000499 = .000554(.987)^8
38	.000598	.014	.000534 = .000598(.986)^8
39	.000648	.015	.000574 = .000648(.985)^8

$$Y = {}_5q_{35}^{[2008]}$$
$$= 1 - {}_5p_{35}^{[2008]}$$
$$= 1 - p_{35}^{[2008]} \cdot p_{36}^{[2008]} \cdot p_{37}^{[2008]} \cdot p_{38}^{[2008]} \cdot p_{39}^{[2008]}$$
$$= 1 - .999565(.999533)(.999501)(.999466)(.999426)$$
$$= .002506$$

B

This is one problem where I can keep ALL the decimal places, since I use the HP-12C calculator. I stored each of the $q_x^{[2008]}$ values into a different register. Then I can recall each one, subtract from 1.0, and multiply them all together.

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