SPRING 2011 EA-1
EXAM SOLUTIONS

2011

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1   This is a typical exam question on yield curves and spot rates. If you understand the definition of those items as well as how to calculate the price of a bond, this is straightforward

Bond price   PV of coupons

$$5,000 = \frac{X}{(1.02)^1} + \frac{X}{(1.04)^2} + \frac{X}{(1.05)^3} + \frac{5,000}{(1.05)^3}$$

The key idea is that each spot rate corresponds to the yield on a zero coupon bond. That is why the present value of a payment in year N is discounted using ONLY the spot rate for the N year period.

$$5000 = X(.9804 + .9246 + .8638) + 5000(.8638)$$

$$X = \frac{5,000 - 4319.19}{2.7688}$$

$$= 245.89$$

Ⓒ

2. This is a very simple question on life annuity calculations. The key point is whether you know the symbols given in the problem:

$$a_{65} = \ddot{a}_{65} - 1 = v\,p_{65} + v^2\,{}_2p_{65} + \ldots$$

$$_{10|}\ddot{a}_{55} = \frac{D_{65}\,\ddot{a}_{65}}{D_{55}} = v^{10}\,{}_{10}p_{55}\,\ddot{a}_{65}$$

$$= v^{10}\,\frac{l_{65}}{l_{55}}\,(1 + a_{65})$$

$$= (1.07)^{-10}\,(.99)^5\,(.95)^5\,(1 + 8.194)$$

$$= .5083\,(.7359)\,(9.194)$$

$$= 3.439$$

$\boxed{E}$

3

This is a typical exam question on nominal versus effective interest rates. You must convert the nominal rates into effective annual rates of return.

Once you have the annual rates of return, you can calculate the accumulated fund value after 17 years.

<u>Nominal rate</u>

7% per year – Discount compounded quarterly $\left[1 - \frac{.07}{4}\right]^{-4} = 1.0732$

8% per year compounded semiannually $\left[1 + \frac{.08}{2}\right]^{2} = 1.0816$

6% per year compounded continuously $e^{.06} = 1.0618$

$X = 1,300 \, (1.0732)^6 (1.0816)^5 (1.0618)^6$

$= 4,213$

(B)

4. The key point of this question is the definition of survival probability under a select and ultimate mortality table. You need to determine the value of the temporary life annuity immediate:

$$a_{[65]:\overline{4}|} = v p_{[65]} + v^2{}_2 p_{[65]} + v^3{}_3 p_{[65]} + v^4{}_4 p_{[65]}$$

$$= \frac{l_{[65]+1}(1.07)^{-1}}{l_{[65]}} + \frac{l_{[65]+2}(1.07)^{-2}}{l_{[65]}} + \frac{l_{[65]+3}(1.07)^{-3}}{l_{[65]}}$$
$$+ \frac{l_{[65]+4}(1.07)^{-4}}{l_{[65]}}$$

Since the select period is only three years, the numerators of the last two terms are actually based on non-select l_x values.

$$a_{[65]:\overline{4}|} = \frac{1}{l_{[65]}}\left[l_{[65]+1}(1.07)^{-1} + \frac{l_{[65]+2}}{(1.07)^2} + \frac{l_{65+3}}{(1.07)^3} + \frac{l_{65+4}}{(1.07)^4} \right]$$

$$= \frac{1}{9,435,643}\left\{ \frac{9,245,121}{1.07} + \frac{9,020,841}{(1.07)^2} + \frac{8,771,863}{(1.07)^3} + \frac{8,511,918}{(1.07)^4} \right\}$$

$$= 3.1978$$

The problem asks for the annual payment of the present value of the annuity is 100,000. Let the payment be represented by P:

$$P(3.1978) = 100,000 \Rightarrow P = 31,271 \quad \text{(B)}$$

5 | This is a typical question on actuarially equivalent benefits. One key point is knowing the formula for a joint and survivor benefit. Another point is that the annuity described in the problem is NOT a typical J+S annuity. You also need to be able to write the death benefit for a reversionary annuity.

Without item III, the benefit described is a typical 50% J+S annuity. The present value of items I+II is

$$12 \left[\ell \left(\ddot{a}_{65:63}^{(12)} \right) + .50 \ell \left(\ddot{a}_{63}^{(12)} - \ddot{a}_{65:63}^{(12)} \right) \right]$$

The second term is the reversionary annuity for the death benefit to Jones after Smith's death. You can use a similar expression for item III, which is a reversionary annuity to Smith after Jones' death:

$$12000 \left(\ddot{a}_{65}^{(12)} - \ddot{a}_{65:63}^{(12)} \right)$$

The value of the original annuity is equal to the present value of the optional annuity (because the benefits are actuarially equivalent):

$$PV \text{ of original} = 12000 \, \ddot{a}_{65}^{(12)}$$
$$= 12 \ell \, \ddot{a}_{65:63}^{(12)} + 6 \ell \left(\ddot{a}_{63}^{(12)} - \ddot{a}_{65:63}^{(12)} \right)$$
$$+ 12000 \left(\ddot{a}_{65}^{(12)} - \ddot{a}_{65:63}^{(12)} \right)$$

(5) To solve for the value of P, you need to set these present values equal to each other. Then collect all terms involving P on the same side

$$12000\ddot{a}_{65}^{(12)} = 6P\ddot{a}_{63}^{(12)} + 6P\ddot{a}_{65:63}^{(12)} + 12000\ddot{a}_{65}^{(12)} - 12000\ddot{a}_{65:63}^{(12)}$$

$$12000\ddot{a}_{65:63}^{(12)} = 6P\left(\ddot{a}_{63}^{(12)} + \ddot{a}_{65:63}^{(12)}\right)$$

$$P = \frac{2000\,\ddot{a}_{65:63}^{(12)}}{\ddot{a}_{63}^{(12)} + \ddot{a}_{65:63}^{(12)}}$$

$$= \frac{2000(9.0)}{12.1 + 9.0}$$
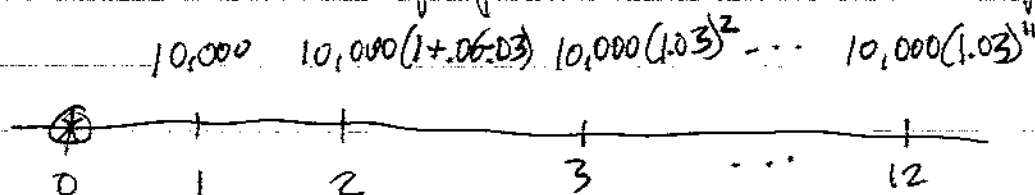
$$= 853 \qquad \textcircled{D}$$

Note that the problem gives the value of $\ddot{a}_{63:65}^{(12)}$, which is the same as the value of $\ddot{a}_{65:63}^{(12)}$.

6 The key point of this problem is interpreting this statement:

"Subsequent payments will be indexed to the excess..."

One interpretation is to simply use the difference between the CPI value and 3%. The more complicated idea would be to use the ratio of (1+ CPI) divided by (1+3%).

I think the simpler approach makes more sense — but it is possible to over-think what is given in the problem. I will determine X using the excess of 6% over 3% each year. It is always a good idea to write down the payments on a time-line diagram:

10,000   10,000(1+.06-.03)   10,000(1.03)^2 ...   10,000(1.03)^{11}

$$X = \frac{10,000}{1.08}\left[1 + \frac{1.03}{1.08} + \cdots + \left(\frac{1.03}{1.08}\right)^{11}\right]$$

The formula in brackets is an annuity due, based on the first payment at time 1. The entire expression must be divided by 1.08, since it is evaluated one year before the first payment. You can calculate the annuity by converting the ratio 1.08/1.03 to a new interest rate — this is typical on the exam.

(6) continued

$$X = (12000/1.08)\, \ddot{a}_{\overline{12}|j} \quad \text{where } 1+j = \frac{1.08}{1.03} \Rightarrow j = 4.85\%$$

$$= (10,000/1.08)(1.0485)\, a_{\overline{12}|4.85\%}$$

Since I use the HP-12C calculator, I always calculate immediate annuity values. I rewrote the annuity due as a product of $a_{\overline{n}|i}$ and $(1+i)$. This avoids silly calculation errors when I forget which type of annuity setting I have on the HP-12C.

$$X = (10,000/1.08)(1.0485)(8.9365)$$
$$= 86,762$$

$$Y = (10,000/1.08)\, \ddot{a}_{\overline{12}|k} \quad \text{where } 1+k = \frac{1.08}{1.01} \Rightarrow k = 6.93\%$$

For Y, the assumed CPI is 4%. The increase in the payments is only 1% per year (4% minus 3%).

$$Y = (10,000/1.08)(1.0693)(7.9721)$$
$$= 78,932$$

$$|X - Y| = 7,830 \qquad \text{(B)}$$

7. This is a fairly typical question on nominal versus effective interest rates. There are several ways to work it, but the shortest technique is to ignore the value of $\ddot{a}_{\overline{2}|}^{(12)}$.

You are given the value of $a_{\overline{2}|}^{(12)}$. You can use this to directly solve for the monthly rate of interest. Then you can calculate the annual effective rate.

$$a_{\overline{2}|}^{(12)} = 1.883280 = \frac{1}{12}\left(a_{\overline{24}|\,j}\right)$$

$$a_{\overline{24}|\,j} = 22.5394 \implies \text{monthly } j = .4868\%$$

using built-in calculator functions

$$(1+i) = (1+j)^{12}$$
$$= 1.0600$$
$$i = 6.0\%$$

$\bigcirc{C}$

8

This is a typical exam question on loans. The key idea is that you must know how to set up the loan amortization schedule. The complicating factor in this problem is the use of two different interest rates.

The first step is determining the monthly loan payment.

$$1,000,000 = P(a_{\overline{180}|j}) + (1.07)^{-15} P(a_{\overline{180}|k})$$

You need to determine the monthly interest rate that are equivalent to 7% for the first 15 years, and 11% for the last 15 years:

$$(1+j)^{12} = 1.07$$
$$j = (1.07)^{\frac{1}{12}} - 1$$
$$= .5654\%$$

$$(1+k)^{12} = 1.11$$
$$k = (1.11)^{\frac{1}{12}} - 1$$
$$= .8735\%$$

$$P = 1,000,000 / \left[a_{\overline{180}|.5654} + (1.07)^{-15} a_{\overline{180}|.8735} \right]$$
$$= 1,000,000 / \left[112.76 + .3624(90.56) \right]$$
$$= 6,869.01$$

(8) continued

looking at the loan amortization schedule for the first 180 payments is not useful, due to the use of two different interest rates. Instead, start with payment number 181:

Payment number	Total Payment	Interest	Principal	Outstanding Loan
180				$6869 \, a_{\overline{180}|.8735}$
181	6869.01	$6869.01(1-v^{180})$	$6869.01 \, v^{180}$	$6869 \, a_{\overline{?}|.8735}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
203	6869.01			$6869 \, a_{\overline{15?}|.8735}$
204	6869.01	$6869.01(1-v^{15?})$		

$$\downarrow$$

$$6869.01\left(1-(1.008735)^{-15?}\right) = 5,115$$

(A)

9. This is a fairly typical problem on bonds. The key point is knowing how to write the formula for the price of a bond:

$$P = Fra_{\overline{n}|k} + Cv^n$$

Since the bond has semiannual coupons, you should convert the yield rates to the same time period

$$(1+k)^2 = 1.04, \qquad (1+j)^2 = 1.05$$
$$k = (1.04)^{\frac{1}{2}} - 1 \qquad j = (1.05)^{\frac{1}{2}} - 1$$
$$= 1.98\% \qquad = 2.47\%$$

$$P = 1000(R/2)a_{\overline{20}|1.98\%} + 1100(1.0198)^{-20}$$

$$P - 95.50 = 1000(R/2)a_{\overline{20}|2.47\%} + 1100(1.0247)^{-20}$$

If you can write down these bond price formulas correctly, then this is simply an algebra problem. You have two equations in two unknowns, and you must solve for the value of R

$$P = R(500)(16.3824) + 1100(.6756) \quad \text{using } 1.98\%$$
$$= 8191.21R + 743.12$$
$$P - 95.50 = R(500)(15.6342) + 1100(.6139) \quad \text{using } 2.47\%$$
$$= 7817.08R + 675.30$$
$$P = 7817.08R + 770.80 = 8191.21R + 743.12$$
$$\therefore R = 27.68/374.13 = 7.40\% \qquad \text{(B)}$$

10 | This is a simplified problem on calculating the modified duration for a portfolio of two bonds. It would have been slightly harder if the yield rate was not 9%, and if the problem did not give you the modified duration for the first bond.

The problem asks for the modified duration. This is defined as the regular duration divided by 1+i:

modified duration = $\bar{d}/(1+i)$

$$\text{regular duration} = \bar{d} = \frac{\sum_{t=1}^{n} t \, v^t R_t}{\sum_{t=1}^{n} v^t R_t}$$

The regular duration is the weighted average duration of the series of payments. The weighting for each payment is the present value of that payment.

The problem gives you the modified duration for the first bond as 6.42. You can easily construct the formulas for the modified duration of each bond, as well as the portfolio

	BOND 1	BOND 2	Portfolio
$\dfrac{\bar{d}}{1+i}$	$\dfrac{\frac{1(90)}{1.09} + \frac{2(90)}{(1.09)^2} + \dots + \frac{10(90+1000)}{(1.09)^{10}}}{\quad}$	$\dfrac{13(1000)}{(1.09)^{13}}$	Σ
	$\dfrac{1}{1.09}\left[\dfrac{90}{1.09} + \dfrac{90}{(1.09)^2} + \dots + \dfrac{90+1000}{(1.09)^{10}}\right]$	$\dfrac{1}{1.09}\left[\dfrac{1000}{(1.09)^{13}}\right]$	Σ

(10) Continued

Look at the formula for the modified duration of bond 1. In the denominator, you have the present value of all payments, divided by 1.09. Since the coupon rate and the yield rate are both 9%, the present value is exactly 1000:

$$\frac{d}{1+i} \text{ for Bond 1} = 6.42 = \frac{\left[\frac{1(90)}{1.09} + \dots + \frac{10(90+1000)}{(1.09)^{10}}\right]}{1000*1.09}$$

$$\therefore \text{numerator} = 6.42(1000*1.09)$$
$$= 6,998$$

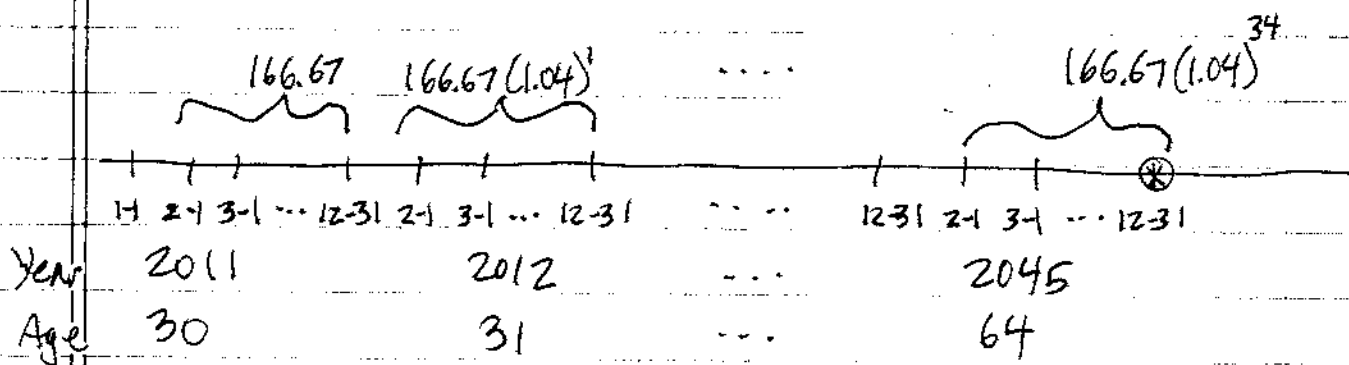Now you can write the formula for the modified duration of the portfolio

$$\frac{d}{1+i} \text{ for portfolio} = X = \frac{\left[\frac{1(90)}{1.09} + \dots + \frac{10(90+1000)}{(1.09)^{10}} + \frac{13(1000)}{(1.09)^{13}}\right]}{1.09(1000 + 1000/(1.09)^{13})}$$

$$= \frac{6,998 + 13,000*.3262}{1.09(1,000 + 1,000*.3262)}$$

$$= \frac{11,238}{1,445}$$

$$= 7.77 \qquad \text{(A)}$$

11   This is a typical exam question with monthly deposits and annual effective interest rates. It is almost the same as 2010 exam question 23!

You must allow for the fact that the monthly deposits increase on an annual basis. The initial deposit amount for 2011 is $5.0\%(40,000)/12 = 166.67$. The key to working the problem is writing down everything on a time line diagram:

There are two ways to work this problem. One approach converts the monthly payments to an annual amount. This payment frequency then matches the annual increases of 4%. And you can work the problem once those items have the same frequency.

The other approach is to leave the monthly deposits, and to convert the series of monthly payments to 12 sets of annual payments, one at 01/31, the next at 02/28 and so on.

(11) Let B represent the accumulated value of the 12/31 monthly payments each year, accumulated to 12/31/2045

$$B = 166.67\left[(1.05)^{34} + (1.04)^{1}(1.05)^{33} + \ldots + (1.04)^{33}(1.05)^{1} + (1.04)^{34}\right]$$

Note that the sum of the exponents for each term is always 34. B represents the accumulated value of each of the twelve series of payments at 01/31, 02/28 etc.

The value of X is calculated by adjusting the twelve accumulated values at 01/31/2045, 02/28/2045, etc. all to 12/31/2045:

$$X = B\left[(1.05)^{11/12} + (1.05)^{10/12} + \ldots + (1.05)^{1/12} + 1\right]$$
$$= B\left(s^{(12)}_{\overline{1}|.05}\right)$$
$$= B\, s_{\overline{12}|j} \quad \text{where } (1+j)^{12} = 1.05 \Rightarrow j = .4074\%$$
$$= B\,(12.2726)$$

To calculate the value of B, the initial summation above will be used. The standard technique is to factor out the first term in the series. This allows the exponents to line up nicely:

$$B = 166.67(1.05)^{34}\left[1 + \left(\frac{1.04}{1.05}\right)^{1} + \ldots + \left(\frac{1.04}{1.05}\right)^{34}\right]$$
$$= 166.67(1.05)^{34}\,\ddot{a}_{\overline{35}|k} \quad \text{where } 1+k = \frac{1.05}{1.04} \Rightarrow k = .9615\%$$

(11) Since I use the HP-12C calculator, I have it set to always calculate immediate annuities, instead of an annuity due. This avoids careless errors, such as forgetting which type of annuity I told the calculator to evaluate!

$$B = 166.67 \, (1.05)^{34} \, \ddot{a}_{\overline{35}|} .9615$$
$$= 166.67 \, (1.05)^{34} \, (1.009615) \, a_{\overline{35}|} .9615$$
$$= 166.67 \, (5.2533)(1.009615)(29.5997)$$
$$= 26,165$$

$$X = B \, (12.2726)$$
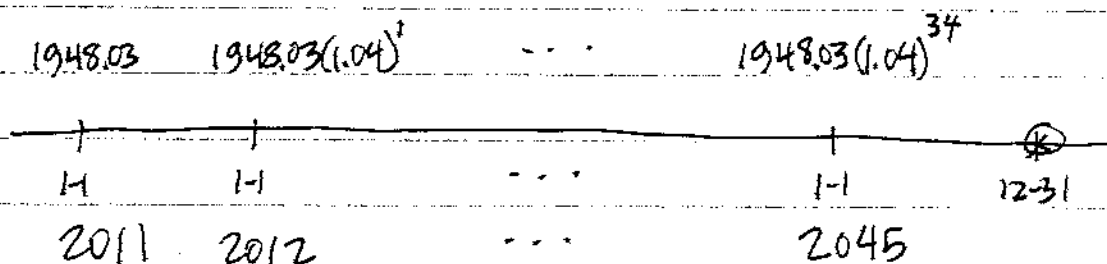$$= 321,117$$

Ⓒ

For the sake of completeness, I will show the solution using the alternate approach that was described earlier. I'll replace the twelve monthly payments with a single payment at 01/01. For the 2011 year, we have

$$01/01 \quad P = 166.67 \, (a_{\overline{12}|}j) \quad \text{where} \, (1+j)^{12} = 1.05 \Rightarrow j = .4074\%$$
$$= 166.67 \, (11.6882)$$
$$= 1,948.03$$

(next page)

(11) I'll rewrite the time line diagram using the new annual payments:

1948.03   1948.03(1.04)^1 $\cdots$ 1948.03(1.04)^{34}

|—| |—| $\cdots$ |—| ⊛
1-1   1-1   1-1   12-31
2011   2012 $\cdots$ 2045

Note that the evaluation point is 12-31-2045. Now you can write an expression for the accumulated value of X:

$$X = 1948.03 \left[(1.05)^{35} + (1.04)^1(1.05)^{34} + \ldots + (1.04)^{34}(1.05)^1 \right]$$

As in the previous expression for B, the sum of the exponents for each term is 35. I will factor out the last term from the series — first to be different

$$X = 1948.03(1.04)^{34}(1.05)^1 \left[\left(\frac{1.05}{1.04}\right)^{34} + \left(\frac{1.05}{1.04}\right)^{33} + \ldots + 1 \right]$$

$$= 1948.03(1.04)^{34}(1.05)^1 \, s_{\overline{35}|k} \quad \text{where } 1+k = \frac{1.05}{1.04} \Rightarrow k = .9615\%$$

$$= 1948.03(3.7943)(1.05)(41.3751)$$

$$= 321,117 \qquad \text{Ⓒ}$$

As expected, the final numerical answer is the same.

12 This is a typical question on the exam involving select and ultimate decrements. The key idea is that you read the decrements going across the table from left to right based on the select age. When you get to the ultimate column, you go down the columns of values.

$$_4q_{[70]+1} = {}_1q_{[70]+1} + {}_1p_{[70]+1}\,q_{[70]+2} + {}_2p_{[70]+1}\,q_{[70]+3}$$
$$+ {}_3p_{[70]+1}\,q_{[70]+4}$$

$$= 1 - {}_4p_{[70]+1}$$
$$= 1 - (1-q_{[70]+1})(1-q_{[70]+2})(1-q_{[70]+3})(1-q_{[70]+4})$$
$$= 1 - (1-q_{[70]+1})(1-q_{73})(1-q_{74})(1-q_{75})$$

I rewrote the q's to reflect the fact that only one falls in the select period. I rewrote the initial expression to make it easier to calculate. The probability that someone dies within the next 4 years is the same as one minus the probability that they live through the next four years.

$$_4q_{[70]+1} = 1 - (1-.070)(1-.082)(1-.085)(1-.088)$$
$$= 1 - .7124$$
$$= .2876$$

Ⓒ

13 This is a typical exam question involving select and ultimate mortality decrements. Unlike the prior problem, this question focuses on various identities, rather than simply "walking through" a table of select and ultimate mortality rates.

In this type of problem, you need to use the information given to reconstruct the table of select and ultimate l_x and d_x values. It may take some "thinking time" to see how the pieces fit together. First step is to use the data on d_x values for the ultimate ages. The second step is to then back into the l_x values:

Age x	d_x	l_x
[0]		
[0]+1		
[0]+2		
3	5000	$105,000 = 100,000 + 5,000$
4	5000	$100,000 = 95,000 + 5,000$
5	5000	$95,000 = 90,000 + 5,000$
6	5000	$90,000$

Now you should write each of the other items given in simplest terms:

$$q_{[0]} = .1667 \implies p_{[0]} = 1 - .1667 \implies l_{[0]+1} = .8333(l_{[0]})$$

(13) continued

$${}_5p_{[1]} = .8000 \Rightarrow \dfrac{l_6}{l_{[1]}} = .8000 \Rightarrow l_{[1]} = \dfrac{l_6}{.80}$$

Now you can solve for the value of $l_{[1]} = 90,000/.80$
$$= 112,500$$

$${}_3p_{[0]+1} = .90\,({}_3p_{[0]}) \Rightarrow \dfrac{l_4}{l_{[0]+1}} = .90\left(\dfrac{l_4}{l_{[1]}}\right) \Rightarrow l_{[1]} = .90\, l_{[0]+1}$$

Now you can substitute the value of $l_{[1]}$ and derive the value of $l_{[0]+1}$:

$$112,500 = .90\, l_{[0]+1} \Rightarrow l_{[0]+1} = 112,500/.90$$
$$= 125,000$$

The final step is to substitute this value into the first relationship given:

$$l_{[0]+1} = .8333\, l_{[0]} \Rightarrow \dfrac{125,000}{.8333} = l_{[0]}$$
$$l_{[0]} = 150,006 \qquad \textcircled{D}$$

14 This is a fairly confusing problem, but there are not many calculations. The key idea is reading it carefully, and interpreting the wording correctly.

When Smith died, the fund received a payment of 1,000,000. You need to calculate the present value of payments to the spouse. The difference between that present value and the fund value of 1,000,000 represents the "charity's interest in the fund".

The spouse is age 65 on January 1. Payments of 70,000 are made at the end of each year for the spouse's lifetime. The present value is based on an immediate life annuity at age 65:

$$PV = 70,000 \, a_{65}$$
$$= 70,000 \, (\ddot{a}_{65} - 1.0)$$
$$= 70,000 \left(\frac{N_{65}}{D_{65}} - 1.0 \right)$$
$$= 70,000 \, (9.6526)$$
$$= 675,682$$

$$X = 1,000,000 - 675,682$$
$$= 324,318$$

(D)

15 This is one of the more confusing exam questions
on stationary population theory. One missing piece
of information is the age range of the population.
Some earlier problems stated that everyone entered
at a particular age, or that everyone exited at a
given age.

My first attempt was to solve for the unknown
entry age — but there is not enough information given
to do that. It appears the only way to work the
problem is to assume that everyone, at _all ages_,
is included.

You can take each statement given in the problem
and write it in terms of the usual terminology and
symbols for population theory. The key to working
the problem is knowing the formula for the average
age at death for those currently age Y and above:

$$\text{average age at death} = Y + \frac{T_Y - T_Z - (Z-Y)l_Z}{l_Y - l_Z}$$

$T_0 = 9800$ (assume entry at birth $\Rightarrow$ age zero)

$l_{25} - l_w = 4(l_0 - l_{25})$ $\Rightarrow$ $l_0 - l_{25} = .25 l_{25}$

l_w is the last age in the mortality table, and equals zero.

(15) continued

Group that is over age 25 is $l_{25} - l_w$, which equals l_{25}. Their average age at death is 66:

Avg age at death

$$66 = 25 + \frac{T_{25} - T_w - (w-25)l_w}{l_{25} - l_w}$$

$$= 25 + \frac{T_{25} - 0 - 0}{l_{25} - 0}$$

$$41 = T_{25}/l_{25} \implies T_{25} = 41 l_{25}$$

Group that is under age 25 is $l_0 - l_{25}$. Their average age at death is 16:

Avg age at death

$$16 = 0 + \frac{T_0 - T_{25} - (25-0)l_{25}}{l_0 - l_{25}}$$

$$= 0 + \frac{9800 - T_{25} - 25 l_{25}}{.25 l_{25}}$$

$$4 l_{25} = 0 + 9800 - 41 l_{25} - 25 l_{25}$$

$$70 l_{25} = 9800$$

$$l_{25} = 140$$

The problem asks for the number who die below age 25 each year, which is $l_0 - l_{25} = .25 l_{25}$

$$X = .25(140) = 35$$

(E)

16　This is a fairly simple question on multiple decrement tables. The key to the solution is knowing the relationship between the single decrement rates and the probability of survival in the multiple decrement table.

For a two decrement table, the probability of survival is calculated based on surviving both decrements each year:

$$p_x^{(T)} = 1 - q_y^{(d)} - q_y^{(w)} = \left[1 - q'^{(d)}_y\right]\left[1 - q'^{(w)}_y\right]$$
$$= (1 - .02)(1 - .04)$$
$$= .9408$$

The problem asks for the probability that the employee receives either a death benefit or a retirement benefit from the plan. Since both benefits require completion of five years of service, the probability X is based on the participant remaining active for three more years:

$$X = {}_3p_{60}^{(T)}$$
$$= (.9408)^3$$
$$= .8327$$

(D)

17 | This is a typical exam question on actuarially equivalent annuities. The basic concept is that the present values of the payments under actuarially equivalent annuities will be equal.

PV of I: $12(1000)\ddot{a}_{55}^{(12)}$

PV of II $12\left[X\ddot{a}_{55:\overline{7}|}^{(12)} + (X-800)\left(\ddot{a}_{55}^{(12)} - \ddot{a}_{55:\overline{7}|}^{(12)}\right)\right]$

$= 12\left[(X-800)\ddot{a}_{55}^{(12)} + 800\,\ddot{a}_{55:\overline{7}|}^{(12)}\right]$

If you can write down the present values correctly, there is nothing difficult remaining. You substitute the annuity values given, and solve for the value of X:

$12(1000)\ddot{a}_{55}^{(12)} = 12\left[(X-800)\ddot{a}_{55}^{(12)} + 800\,\ddot{a}_{55:\overline{7}|}^{(12)}\right]$

$1000 = X-800 + 800\left(\dfrac{\ddot{a}_{55:\overline{7}|}^{(12)}}{\ddot{a}_{55}^{(12)}}\right)$

$X = 1800 - 800\left(\dfrac{5.50}{11.33}\right)$

$= 1,412$ (D)

A "sanity check" is to look at the annuity payments for the second annuity. They exceed 1,000 /mo in the beginning, and are less than 1,000/mo at the end. This is the expected result.

18. This is a typical exam question on probability definitions. You need to write down expressions for each item you are given. You may have to think a while to see how you can derive a value for X.

$$.33\,(l_{30} - l_{70}) = l_{30} - l_{50}$$

$$_{20}q_{30} = .20 \Rightarrow \frac{l_{30} - l_{50}}{l_{30}} = .20$$

$$X = _{20}p_{50} = \frac{l_{70}}{l_{50}}$$

You can use the second relationship to get a formula for the ratio of l_{30} versus l_{50}:

$$\frac{l_{30} - l_{50}}{l_{30}} = .20 \Rightarrow 1 - \frac{l_{50}}{l_{30}} = .20 \quad \frac{l_{50}}{l_{30}} = .80$$

If you divide both sides of the first expression by l_{50}, then you'll get a result with $\frac{l_{70}}{l_{50}}$, which is X:

$$.33\,\frac{(l_{30} - l_{70})}{l_{50}} = \frac{l_{30} - l_{50}}{l_{50}} \Rightarrow .33\left(\frac{l_{30}}{l_{50}} - X\right) = \frac{l_{30}}{l_{50}} - 1$$

Now substitute the value of $l_{30}/l_{50} = 1/.80 = 1.25$

$$.33\,(1.25 - X) = 1.25 - 1 \Rightarrow X = 1.25 - .758 = .492 \quad \textcircled{C}$$

(15) This is a very tough problem on select and ultimate decrements. You have to be very careful in going through lots of algebra — and it is very easy to write down the wrong formulas at the very start of the problem.

You are told that the expected withdrawal for 2012 equal 120. You need to analyze the data on the population to determine how many actives you have at 1-1-2012:

01-01-2011				01-01-2012		
Service	Age 40	Age 41		Service	Age 41	Age 42
0	200			1	$200\,p_{[40]}$	
1		150		2		$150\,p_{[40]+1}$
2+	900			3+	$900\,p_{40}$	

The next step is writing down values for the select and ultimate q_x values at ages 40, 41 and 42:

$$q_x^{(\omega)} = .100 - .003(x-40) \Rightarrow q_{40}^{(\omega)} = .100$$
$$q_{41}^{(\omega)} = .100 - .003 = .097$$
$$q_{42}^{(\omega)} = .100 - .003(2) = .094$$

$$q_{[40]+s}^{(\omega)} = q_{[40]}^{(\omega)} - .02s \text{ for } s = 0,1 \Rightarrow q_{[40]+1}^{(\omega)} = q_{[40]}^{(\omega)} - .02$$
$$= X - .02$$

(19) continued

Now you should write the formula for the expected exits in 2012:

$$120 = 200 \, p_{[40]} q_{[40]+1}^{(w)} + 900 \, p_{40} q_{41}^{(w)} + 150 \, p_{[40]+1} q_{42}^{(w)}$$
$$= 200(1 - q_{[40]}^{(w)}) q_{[40]+1}^{(w)} + 900(1 - q_{40}^{(w)}) q_{41}^{(w)} + 150(1 - q_{[40]+1}^{(w)}) q_{42}^{(w)}$$

Now you can substitute the values of q based on the data given in the problem

$$120 = 200(1-x)(x-.02) + 900(1-.100)(.097) + 150(1-x+.02)(.094)$$
$$.60 = 1(1-x)(x-.02) + 4.5(.90)(.097) + .75(1.02-x)(.094)$$
$$= x - .02 - x^2 + .02x + .39285 + .07191 - .0705x$$
$$0 = -x^2 + (1.02x - .0705x) + (-.02 + .39285 + .07191 - .60)$$
$$= x^2 - .9495x + .15524$$

This the easilty solved using the quadratic equation. There will be two answers, but one will give a value of x that exceeds .50, which the problem states is an invalid result.

$$x = (-b \pm \sqrt{b^2 - 4ac})/2a$$
$$= \frac{.9495 \pm \sqrt{(.9495)^2 - 4(1)(.15524)}}{2(1)}$$
$$= \frac{.9495 \pm .52971}{2}$$
$$= .2099 \qquad \boxed{E}$$

20 | This is a typical exam question on actuarially equivalent benefits — even though the problem does not explicitly say that, it is implied. The key is writing down the present values of each form, and then setting them equal to each other.

$$PV \text{ of } I \qquad 15,000 \, \ddot{a}_{65}^{(12)}$$

$$PV \text{ of } II \qquad 13,500\left[\ddot{a}_{65}^{(12)} + .50\left(\ddot{a}_y^{(12)} - \ddot{a}_{65:y}^{(12)}\right)\right]$$

$$PV \text{ of } III \qquad P\left[\ddot{a}_{65}^{(12)} + .75\left(\ddot{a}_y^{(12)} - \ddot{a}_{65:y}^{(12)}\right)\right]$$

$$15,000 \, \ddot{a}_{65}^{(12)} = 13,500\left[\ddot{a}_{65}^{(12)} + .50\left(\ddot{a}_y^{(12)} - \ddot{a}_{65:y}^{(12)}\right)\right]$$

$$15,000 \, \ddot{a}_{65}^{(12)} = P\left[\ddot{a}_{65}^{(12)} + .75\left(\ddot{a}_y^{(12)} - \ddot{a}_{65:y}^{(12)}\right)\right]$$

If you look at these two equations, it appears there are too many unknown values to solve for the value of P. But there is a trick you can use — first divide both sides by $\ddot{a}_{65}^{(12)}$:

$$15,000 = 13,500\left[1 + .50\left(\ddot{a}_y^{(12)} - \ddot{a}_{65:y}^{(12)}\right)/\ddot{a}_{65}^{(12)}\right]$$

$$15,000 = P\left[1 + .75\left(\ddot{a}_y^{(12)} - \ddot{a}_{65:y}^{(12)}\right)/\ddot{a}_{65}^{(12)}\right]$$

The second part of the solution is to simply replace the ratio of the annuity values with a variable. You really don't care about the value of each individual annuity.

(20) Continued

$$15,000 = 13,500 \left[1 + .50R \right]$$

$$15,000 = P \left[1 + .75R \right]$$

Now you can derive a value of R from the first equation, and solve for the value of P using the second equation.

$$15,000/13,500 = 1 + .50R \Rightarrow R = 2 \left[\frac{15,000}{13,500} - 1 \right]$$
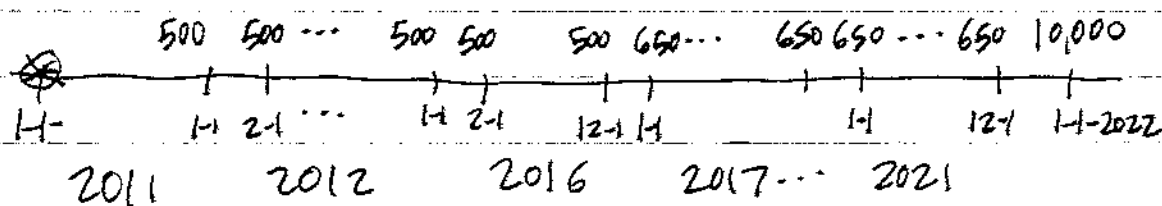
$$= .2222$$

$$P = \frac{15,000}{1 + .75R}$$

$$= 12,857 \qquad \textcircled{D}$$

This problem solution technique has been tested almost every year on recent exams!

21 | This is a straightforward question on present value calculations involving interest rates. You have monthly payments, but an annual interest rate. You need the payment frequency and the interest rate compounding period to be the same. The typical choice would be to determine the equivalent monthly interest rate to match the payment frequency:

$$(1+j)^{12} = 1.07 \Rightarrow j = (1.07)^{1/12} - 1 = .5654\%$$

To avoid silly errors, it is useful to write down all of the payments on a time line diagram:

One minor trick is that the first payment starts one year after the date to determine the present value.

$$PV = (1.07)^{-1} \left[6000\, \ddot{a}_{\overline{5}|.07}^{(12)} + (1.07)^{-5}(7,800)\, \ddot{a}_{\overline{5}|.07}^{(12)} \right] + (1.07)^{-11}(10,000)$$

I find it easiest to work this problem using monthly payments with a monthly interest rate:

$$PV = (1.07)^{-1} \left[500(1.005654)\, a_{\overline{60}|.5654} + (1.07)^{-5}(650)(1.005654)\, a_{\overline{60}|.5654} \right]$$
$$+ (1.07)^{-11}(10,000)$$

(21) continued
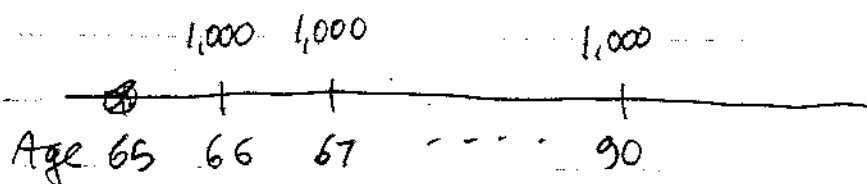
Instead of valuing the payments using $\ddot{a}_{\overline{60}|.5654}$, I wrote the annuity as $1.005654(a_{\overline{60}|.5654})$. I do this to avoid silly errors when calculating annuities on the HP-12C calculator. I have the calculator set to always calculate immediate annuities.

$$PV = .9346[500(1.005654)(50.7617) + .7130(650)(1.005654)(50.7617)]$$
$$+.4751(10,000)$$
$$= 45,965 + 4,751$$
$$= 50,716 \qquad \qquad \textcircled{B}$$

22   This is a typical present value calculation for the exam. The key is to write down the payments on a time line diagram first. Then you can write the formula for the present value of the annuity:

$$1{,}000 \quad 1{,}000 \quad\quad\quad 1{,}000$$

Age 65   66   67   - - - -   90

The annuity is a 25 year temporary annuity with payments for Smith's life:

$$Y = 1000 \, a_{65:\overline{25}|}$$
$$= 1000\left(v \, p_{65} + v^2 \, {}_2p_{65} + \ldots + v^{25} \, {}_{25}p_{65}\right)$$

The problem states that mortality rates double at ages 75 and above. Prior to age 75, $q_x = 5\%$ and above age 75, $q_x = 10\%$.

$$Y = 1000\left(\frac{.95}{1.05} + \left(\frac{.95}{1.05}\right)^2 + \ldots + \left(\frac{.95}{1.05}\right)^{10} + \frac{.90(.95)^{10}}{(1.05)^{11}} + \ldots + \frac{(.90)^{15}(.95)^{10}}{(1.05)^{25}}\right)$$

$$= 1000\left[\left(\frac{.95}{1.05}\right)^1 + \ldots + \left(\frac{.95}{1.05}\right)^{10} + \left(\frac{.95}{1.05}\right)^{10}\left\{\left(\frac{.90}{1.05}\right)^1 + \ldots + \left(\frac{.90}{1.05}\right)^{15}\right\}\right]$$

You can value this annuity as the sum of two annuities at two different interest rates for 10 years, and then 5 years.

$$Y = 1000\left[a_{\overline{10}|j} + (.95/1.05)^{10} \, a_{\overline{15}|k}\right] \quad \text{where} \quad 1+j = 1.05/.95 = 1.10526$$
$$\text{and} \quad 1+k = 1.05/.90 = 1.16667$$

(22) continued

$$Y = 1000 \left[a_{\overline{10}|10.526\%} + (1.10526)^{-10} a_{\overline{15}|16.667\%} \right]$$
$$= 1000 \left[6.0081 + .3676(5.4058) \right]$$
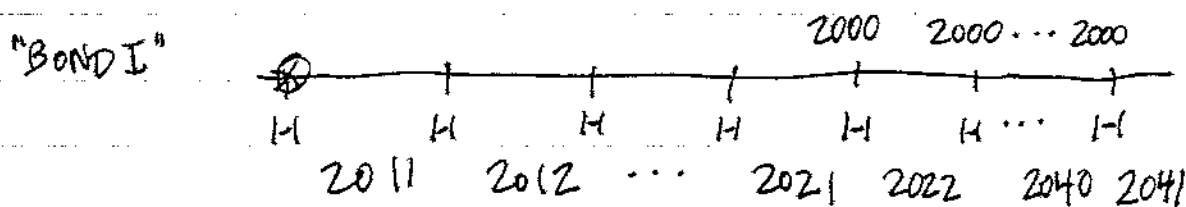$$= 7,395$$

Ⓒ

23 | This is a typical exam problem on the calculation of duration values. The problem asks for the modified duration. This is defined as the regular duration divided by $(1+i)$:

regular duration

$$\bar{d} = \frac{\sum\limits_{t=1}^{n} t\, v^t R_t}{\sum\limits_{t=1}^{n} v^t R_t}$$

modified duration $= \bar{d} / (1+i)$

The regular duration is the weighted average time until payment for a series of payments. The weighting factor for each payment is the present value of that payment.

The problem gives you two "bonds". You need to construct the formula for the modified duration for each "bond" separately. Then you can combine the results to calculate the modified duration of the portfolio.

"BOND I"

"Bond I" has 20 payments, starting 11 years from now.

23 | (continued)

"Bond I" $\bar{d} = \dfrac{11(2000)v^{11} + 12(2000)v^{12} + \ldots + 30(2000)v^{30}}{2000(v^{11} + v^{12} + \ldots + v^{30})}$

regular duration

You need to calculate the increasing annuity in the numerator separately, which takes some algebra:

$$N = 2000\left[11v^{11} + 12v^{12} + \ldots + 30v^{30}\right]$$

$$vN = 2000\left[0 + 11v^{12} + \ldots + 29v^{30} + 30v^{31}\right] \quad \text{multiply by } v$$

$$N(1-v) = 2000\left[11v^{11} + v^{12} + \ldots v^{30} - 30v^{31}\right] \quad \text{subtract}$$

$$N = \dfrac{2000}{1-v}\left[10v^{11} + (v^{11} + v^{12} + \ldots + v^{30}) - 30v^{31}\right]$$

$$= \dfrac{2000}{iv}\left[10v^{11} + (a_{\overline{30}} - a_{\overline{10}}) - 30v^{31}\right]$$

$$= \dfrac{2000}{.08/1.08}\left[10(1.08)^{-11} + 11.2578 - 6.7101 - 30(1.08)^{-31}\right]$$

$$= 27{,}000\left[4.2888 + 11.2578 - 6.7101 - 2.7605\right]$$

$$= 164{,}053$$

"Bond II"   This zero coupon bond has a

regular duration   single payment in 5 years

$$\bar{d} = \dfrac{5(10{,}000)v^{5}}{10{,}000\,v^{5}}$$

(23i) continued

Now you can calculate the regular duration for the entire portfolio. You must add the numerators and denominators previously calculated:

Portfolio
regular
duration

$$\bar{d} = \frac{164,053 + 5(10,000)(1.08)^{-5}}{2,000\left(a_{\overline{30}|.08} - a_{\overline{10}|.08}\right) + 10,000(1.08)^{-5}}$$

$$= \frac{164,053 + 34,029}{9,095 + 6,806}$$

$$= 198,082 / 15,901$$

$$= 12.46$$

modified
duration

$$\frac{\bar{d}}{1+i} = \frac{12.46}{1.08}$$

$$= 11.53$$

Ⓒ

24 This is a typical exam question on the various identities associated with the mortality laws. You must know most of the identities for the assumption of uniform distribution of decrements for each year of age.

This problem gives you values for several items, plus l_{43}. You can use each item to derive the value of l_x at the next earlier age

<u>Given</u>

$._5 q_{40.4} = .025$

$._9 p_{41} = .955$

$\mu_{42.2} = .050$

$l_{43} = 100,000$

<u>Identity – UDD</u>

$$y q_{x+t} = \frac{y(q_x)}{1 - t(q_x)}$$

$$t p_x = 1 - t(q_x)$$

$$\mu_{x+t} = \frac{q_x}{1 - t(q_x)}$$

Start with $\mu_{42.2} = .05 = \dfrac{q_{42}}{1 - .2(q_{42})}$

$q_{42} = .05 - .01(q_{42})$
$= .05/1.01 = .0495$

$\therefore p_{42} = .9505 = l_{43}/l_{42}$

$l_{42} = l_{43}/.9505 = 100,000/.9505 = 105,208$

Now go to the next lower age

$._9 p_{41} = .955 = 1 - .9(q_{41})$

$.9(q_{41}) = 1 - .955$

$q_{41} = .050$

$\therefore p_{41} = .950 = l_{42}/l_{41}$

(24) continued

$$\ell_{41} = \ell_{42}/.950 = 110,746 = 105,208/.950$$

$$.5q_{40.4} = .025 = \frac{.5(q_{40})}{1-.4(q_{40})}$$

$$.025 - .010 q_{40} = .5 q_{40}$$

$$q_{40} = .025/.51 = .049$$

$$\therefore p_{40} = \ell_{41}/\ell_{40} = .951$$

$$\ell_{40} = \ell_{41}/.951 = 110,746/.951 = 116,454$$

Ⓑ

25 | This is a fairly difficult loan amortization question. You must carefully write down the amortization schedule to identify what the interest payments are each year:

Payment number	Principal amount	Interest amount	Outstanding Loan
0			L
1	$100(1)$	$.06L$	$L-100$
2	$100(2)$	$.06(L-100)$	$L-300$
3	$100(3)$	$.06(L-300)$	$L-600$
⋮	⋮	⋮	⋮
n	$100(n)$	$.06(L-\dfrac{100(n)(n-1)}{2})$	$L-\dfrac{100(n+1)(n)}{2}$

The key point of the problem is recognizing that you can construct formulas for the outstanding loan and the interest amount. You can determine the amount of the original loan — it is the sum of all the principal payments over the 20 years:

$$L = 100(1) + 100(2) + ... + 100(20)$$
$$= 100(20)(21)/2$$
$$= 21,000$$

You need to write the last 2 lines of the amort. schedule:

19	$100(19)$	$.06(L-50(19)(18))$	$L-50(19)(20)$
20	$100(20)$	$.06(L-50(20)(19))$	zero

(25) continued

Now you can write the formula for the present value of the interest payments. I found it better to write this using L as the original loan amount

$$X = .06\left[\frac{L}{1.06} + \frac{L-100}{(1.06)^2} + \frac{L-300}{(1.06)^3} + \frac{L-600}{(1.06)^4} + \ldots + \frac{L-50(18)(19)}{(1.06)^{19}} + \frac{L-50(19)(20)}{(1.06)^{20}}\right]$$

The easy way to solve for X is to multiply the entire formula by $(1.06)^{-1}$, and subtract that from the original formula:

$$\frac{X}{1.06} = .06\left[0 + \frac{L}{(1.06)^2} + \frac{L-100}{(1.06)^3} + \frac{L-300}{(1.06)^4} + \ldots + \ldots + \frac{L-50(18)(19)}{(1.06)^{20}} + \frac{L-50(19)(20)}{(1.06)^{21}}\right]$$

$$X\left(1 - \frac{1}{1.06}\right) = .06\left[\frac{L}{1.06} - \frac{100}{(1.06)^2} - \frac{200}{(1.06)^3} - \frac{300}{(1.06)^4} - \ldots - \frac{1900}{(1.06)^{20}} - \frac{L-19,000}{(1.06)^{21}}\right]$$

$$X\left(\frac{.06}{1.06}\right) = .06\left[\frac{21,000 - 100(I\ddot{a}_{\overline{19}|}.06)}{1.06} - \frac{2,000}{(1.06)^{21}}\right]$$

$$X = 1.06\left[\frac{21,000 - 100(I\ddot{a}_{\overline{19}|}.06)}{1.06} - \frac{2,000}{(1.06)^{21}}\right]$$

$$= 21,000 - 100\left(\frac{\ddot{a}_{\overline{19}|}.06 - 19v^{19}}{.06}\right) - \frac{2,000}{(1.06)^{20}}$$

$$= 21,000 - 1666.67\left(1.06\, a_{\overline{19}|}.06 - 19(1.06)^{-19}\right) - 2,000(1.06)^{-20}$$

$$= 21,000 - 1666.67(11.8276 - 6.2797) - 624$$

$$= 11,130 \qquad \qquad Ⓓ$$

This problem is IDENTICAL to 2006 exam problem 5!

26 This is a typical exam question on bonds. The key idea is that the amortized value is the same as the purchase price of the bond at that point in time.

This problem does not tell you the total number of coupons, or the redemption date. But you can still set up formulas for the bond price at 06/30/11 and 12/31/11. Let j be the semi-annual yield rate, so $(1+j)^2 = 1+X$.

06-30-11 Price $939.33 = 1000(.07/2)(1+a_{\overline{n}|j}) + 1000(1+j)^{-n}$

12-31-11 Price $943.78 = 1000(.07/2)(1+a_{\overline{n-1}|j}) + 1000(1+j)^{-(n-1)}$

$$= 35\ddot{a}_{\overline{n}|j} + 1000(1+j)^{-n-1}$$

Now multiply the 06-30 price formula by $(1+j)$, and you can subtract the two formulas:

$$(1+j)(939.33) = 35[(1+j) + \ddot{a}_{\overline{n}|j}] + 1000(1+j)^{-(n-1)}$$

$$943.78 = 35[\quad \ddot{a}_{\overline{n}|j}] + 1000(1+j)^{-(n-1)}$$

$$939.33j - 4.45 = 35(1+j)$$
$$904.33j = 39.45$$
$$j = 4.36\%$$

$$1+X = (1+j)^2 = 1.0891$$
$$X = 8.91\% \qquad \text{(E)}$$

27 This is a typical exam question on both time weighted and dollar weighted return calculations. The key to this problem is writing down all the cash flows and market values on a time line diagram.

For the time weighted returns, you need the market values both before and after every cash flow. I'll use MV_B for the market value before a cash flow, and MV_A for the market value after a cash flow.

You are given the dollar weighted return as 7.0%. You can use this to derive the value of the benefit payments. I'll accumulate the value of every cash flow, plus the beginning market value using simple interest. The result equals the final market value.

$$50,000\left(1+.07\left(\tfrac{12}{12}\right)\right) - P\left(1+.07\left(\tfrac{9}{12}\right)\right) + (17,000-P)\left(1+.07\left(\tfrac{6}{12}\right)\right)$$
$$+ (55,000-P)\left(1+.07\left(\tfrac{3}{12}\right)\right) = 65,000$$

$$50,000(1.07)+17,000(1.035)+55,000(1.0175)-65,000 = P\left[1.0525+1.035+1.0175\right]$$
$$P = \frac{62,057.50}{3.1050} = 19,986.31$$

To calculate the time-weighted return you must develop the market values before (and after) the cash flows at 04-01, 07-01 and 10-01.

(27) continued

MV$_B$	60,000	45,000	40,000	65,000
Cash Flow	-19,986	-2,986	35,014	

Date	01-01	04-01	07-01	10-01	12-31
MV$_A$	50,000	40,014	42,014	75,014	

The time weighted return is calculated by using ratios of the market values, between the cash flows:

MV$_b$ date   03-31   06-30   09-30   12-31

$$1+X = \left(\frac{60,000}{50,000}\right)\left(\frac{45,000}{40,014}\right)\left(\frac{40,000}{42,014}\right)\left(\frac{65,000}{75,014}\right)$$

MV$_A$ date   01-01   04-01   07-01   10-01

$$1+X = (1.20)(1.1246)(.9521)(.8665)$$

$$= 1.1133$$

$$X = 16.33\%$$

(D)

28 | This is a typical question on the force of mortality and multiple decrement tables. It is similar to 2000 exam problem 20, and almost identical to 1996 exam problem 1.

There are two (or more) solutions for this problem, but they all give identical results. The main reason is that each of the three decrements conforms to DeMoivre's law, with a constant number of exits for each year of age.

The definition of the multiple decrement force of mortality is a bit technical:

$$\mu_x^{(k)} = \frac{-1}{l_x^{(T)}} \left[\frac{d}{dx} l_x^{(k)} \right]$$

I'll show a less theoretical solution first, which matches what I did for 1996 #1. You are given l_x formulas for three decrement tables. In a multiple decrement table, you have the following definitions

$$q_x^{(T)} = q_x^{(1)} + q_x^{(2)} + q_x^{(3)} + \ldots \quad \text{in general}$$

$$d_x^{(T)} = d_x^{(1)} + d_x^{(2)} + d_x^{(3)} \quad \text{with only three decrements}$$

$$l_x^{(k)} = d_x^{(k)} + d_{x+1}^{(k)} + d_{x+2}^{(k)} + \ldots$$
$$l_x^{(T)} = l_x^{(1)} + l_x^{(2)} + l_x^{(3)}$$

(28) continued

you can write down the l_x values for the ages 49, 50 and 51. This will be useful in "calculating" the force of mortality at age 50

Age x	$l_x^{(1)}$	$l_x^{(2)}$	$l_x^{(3)}$	$l_x^{(T)}$
49	153	244	355	752
50	150	240	350	740
51	147	236	345	728
d_x	3	4	5	12

The idea of $\dfrac{d}{dx} l_x^{(k)}$ is the rate of decrease in $l_x^{(k)}$ at a given age x. All three of the l_x tables use De Moivre's law, so the number of deaths is a constant each age. That means that the derivative of the l_x is also constant.

$$\mu_{50}^{(2)} = \frac{-1}{l_{50}^{(T)}} \left[\frac{d\, l_{50}^{(2)}}{dx} \right]$$
$$= \frac{-1}{740} (-4)$$
$$= .0054$$

(B)

(28) continued

Another solution that gives the same result is use of the approximation formula Jordan 14.26:

$$\mu_x^{(K)} \doteq \frac{d_{x-1}^{(K)} + d_x^{(K)}}{2 l_x^{(T)}}$$

Instead of using the derivative for $l_x^{(K)}$, this approximates the value by using two years' worth of exits. This gives exactly the same result of .0054.

A more theoretical approach is to use the formulas given for the l_x tables:

$$l_x^{(1)} = 3(100-x)$$
$$l_x^{(2)} = 4(110-x) \qquad \Rightarrow \quad \frac{d}{dx} l_x^{(2)} = 0 - 4 = -4$$
$$l_x^{(3)} = 5(120-x)$$

$$l_x^{(T)} = l_x^{(1)} + l_x^{(2)} + l_x^{(3)}$$
$$= 1340 - 12x$$

$$\mu_x^{(2)} = \frac{-1}{l_x^{(T)}} \left[\frac{d\, l_x^{(2)}}{dx} \right]$$
$$= \frac{-1}{1340-12x} (-4)$$
$$= 4/(1340-12x)$$
$$\mu_{50}^{(2)} = 4/(1340-600) = .0054 \qquad Ⓑ$$

29 | This is a typical exam question on actuarially equivalent benefits. You must write down the formula for both annuities very carefully. Then you can set them equal to each other and solve for X. One point of the problem is that you'll have to derive some missing items to calculate both present values.

PV of normal form $\quad X\left(\ddot{a}_{\overline{10}|.07} + v^{10}\,{}_{10}p_{55}\,\ddot{a}_{65}\right)$

PV of optional form $\quad 10{,}000\,\ddot{a}_{55:\overline{5}|} + 7{,}500\left(\ddot{a}_{55:\overline{10}|} - \ddot{a}_{55:\overline{5}|}\right)$
$$+ 5{,}000\left(\ddot{a}_{55} - \ddot{a}_{55:\overline{10}|}\right)$$

There is another way to write the optional form annuity. Think in terms of layers of benefits. You have 5,000 for life, plus 2,500 for 10 years, plus 2,500 for the first 5 years:

PV of optional form $\quad 5{,}000\,\ddot{a}_{55} + 2{,}500\,\ddot{a}_{55:\overline{10}|} + 2{,}500\,\ddot{a}_{55:\overline{5}|}$

You are missing two items: ${}_{10}p_{55}$ and $\ddot{a}_{55:\overline{10}|}$. You can derive ${}_{10}p_{55}$ based on the annuities given. The key idea is that you can put together two temporary 5 year annuities to equal the 10 year temporary annuity.

$$\ddot{a}_{55} = 1 + v\,p_{55} + v^2\,{}_2p_{55} + \dots + v^{10}\,{}_{10}p_{55} + v^{11}\,{}_{11}p_{55} + \dots$$
$$= \ddot{a}_{55:\overline{10}|} + v^{10}\,{}_{10}p_{55}\,\ddot{a}_{65}$$
$$= \ddot{a}_{55:\overline{5}|} + v^5\,{}_5p_{55}\,\ddot{a}_{60:\overline{5}|} + v^{10}\,{}_{10}p_{55}\,\ddot{a}_{65}$$

(29) Continued

Now there is another probability to derive: $_5p_{ss}$.
You can use the annuities given to do this in a
similar way as the previous formulas:

$$\ddot{a}_{ss} = \ddot{a}_{ss:\overline{5}|} + {}_{5|}\ddot{a}_{ss}$$
$$= \ddot{a}_{ss:\overline{5}|} + v^5 {}_5p_{ss} \ddot{a}_{60}$$
$${}_5p_{ss} = \frac{\ddot{a}_{ss} - \ddot{a}_{ss:\overline{5}|}}{v^5 \ddot{a}_{60}}$$
$$= \frac{(1.07)^5 (11.2751 - 4.3122)}{10.2758}$$
$$= .9504$$

Now you can use the earlier formulas to derive $_{10}p_{ss}$:

$$\ddot{a}_{ss} = \ddot{a}_{ss:\overline{5}|} + v^5 {}_5p_{ss} \ddot{a}_{60:\overline{5}|} + v^{10} {}_{10}p_{ss} \ddot{a}_{65}$$
$$\ddot{a}_{ss} - \ddot{a}_{ss:\overline{5}|} - v^5 {}_5p_{ss} \ddot{a}_{60:\overline{5}|} = v^{10} {}_{10}p_{ss} \ddot{a}_{65}$$
$${}_{10}p_{ss} = \frac{\ddot{a}_{ss} - \ddot{a}_{ss:\overline{5}|} - v^5 {}_5p_{ss} \ddot{a}_{60:\overline{5}|}}{v^{10} \ddot{a}_{65}}$$
$$= \frac{(1.07)^{10} [11.2751 - 4.3122 - (1.07)^{-5}(.9504)(4.2707)]}{9.1301}$$
$$= .8767$$

Now you can go back to the original formula and
calculate the value of x.

(29) continued

$$X\left(\ddot{a}_{\overline{10}|.07} + v^{10}{}_{10}p_{55}\,\ddot{a}_{65}\right) = 5000\,\ddot{a}_{55} + 2,500\,\ddot{a}_{55:\overline{10}|} + 2,500\,\ddot{a}_{55:\overline{5}|}$$

One more annuity derivation is needed: $\ddot{a}_{55:\overline{10}|}$
is not yet defined.

$$\ddot{a}_{55:\overline{10}|} = \ddot{a}_{55:\overline{5}|} + v^5\,{}_5p_{55}\,\ddot{a}_{60:\overline{5}|}$$
$$= 4.3122 + (1.07)^5(.9504)(4.2707)$$
$$= 7.2060$$

$$X = \frac{5,000\,\ddot{a}_{55} + 2,500\,\ddot{a}_{55:\overline{10}|} + 2,500\,\ddot{a}_{55:\overline{5}|}}{\ddot{a}_{\overline{10}|.07} + v^{10}{}_{10}p_{55}\,\ddot{a}_{65}}$$
$$= \frac{5,000(11.2751) + 2,500(7.2060) + 2,500(4.3122)}{1.07(a_{\overline{10}|.07}) + (1.07)^{-10}(.8767)(9.1301)}$$

I wrote the formula using an immediate annuity
for a specific reason. I use the HP-12C calculator,
which allows you to calculate either an annuity due,
or an annuity immediate. I avoid making silly
calculation errors by always calculating immediate
annuity values on the HP-12C.

$$X = \frac{56,376 + 18,015 + 10,781}{7.5152 + 4.0691}$$
$$= 7,352 \qquad \text{Ⓐ}$$

30 | This is a typical exam question on life insurance. You need to write out the formula for the premium and express it in simplest terms. That will allow you to use the factors given in the problem.

PV of insurance benefit = PV of premiums

$$100,000 \, A^{1}_{50:\overline{3}|} = X$$

$$X = 100,000 \left(v \, q_{50} + v^2 \, p_{50} \, q_{51} + v^3 \, p_{50}(p_{51}) \, q_{52} \right)$$

$$= 100,000 \left(v \, q_{50} + v^2 \, (_1|q_{50}) + v^3 \, (_2|q_{50}) \right)$$

You need to derive the value of q_{50} based on the information given:

$$_2p_{50} = p_{50}(p_{51})$$
$$= (1 - q_{50})(1 - q_{51}) = .9955$$

$$_1|q_{50} = (1.08)^1 q_{50}$$
$$= p_{50} \, q_{51} = (1 - q_{50}) q_{51}$$
$$(1 - q_{50}) q_{51} = 1.08 \, q_{50}$$
$$q_{51} = 1.08 \, q_{50} / (1 - q_{50}) \qquad \text{now substitute in prior formula}$$

$$.9955 = (1 - q_{50})(1 - q_{51})$$
$$= (1 - q_{50})\left(1 - \frac{1.08 \, q_{50}}{1 - q_{50}}\right)$$

(30) continued

$$.9955 = 1 - q_{50} - 1.08 q_{50}$$
$$2.08 q_{50} = 1 - .9955$$
$$q_{50} = .0022$$

Now you can go back to the original equation and solve for the value of X:

$$X = 100,000 \left(v^1 q_{50} + v^2 (_1| q_{50}) + v^3 (_2| q_{50}) \right)$$
$$= 100,000 \left(\frac{q_{50}}{1.06} + \frac{1.08 \, q_{50}}{(1.06)^2} + \frac{(1.08)^2 q_{50}}{(1.06)^3} \right)$$
$$= 100,000 \left(\frac{q_{50}}{1.06} \right) \left[1 + \frac{1.08}{1.06} + \left(\frac{1.08}{1.06} \right)^2 \right]$$
$$= 100,000 \frac{(.0022)}{1.06} \, s_{\overline{3}|j} \qquad \text{where } 1+j = \frac{1.08}{1.06} \Rightarrow j = 1.887\%$$
$$= 100,000 (.0020)(3.0570)$$
$$= 624$$

(A)

31 | This problem looks familiar — it is identical to problem 17 from the 2009 exam! In the real world, different actuaries may calculate the weighted average in different ways. But there is very little information given in the problem, which means there is only one way to work this problem.

You must calculate a simple average, based on the retirement decrements given. You must figure out what portion of someone (who is under age 62) will retire at each future age:

Age x	Probability of surviving and retiring at age x
62	$100\% (40\%) = .40$
63	$(1-.40)(25\%) = .15$
64	$(1-.40)(1-.15)(.25) = .1125$
65	$(1-.40)(1-.15)(1-.25)(100\%) = .3375$

One thing to check is that the sum of all percentages adds up to 100% — if it does not, then there is an error somewhere!

The final step is the weighted retirement age:

$$X = 62(.40) + 63(.15) + 64(.1125) + 65(.3375)$$
$$= 63.39$$

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