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SPRING 2012 EA-1 EXAM SOLUTIONS

Revision History:
02/20/16 Revised solution for problems 10 and 14

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1 This is a simple question on mortality laws. You only need to know one definition under the assumption of uniform distribution of deaths.

UDD $_t q_x = t(q_x)$

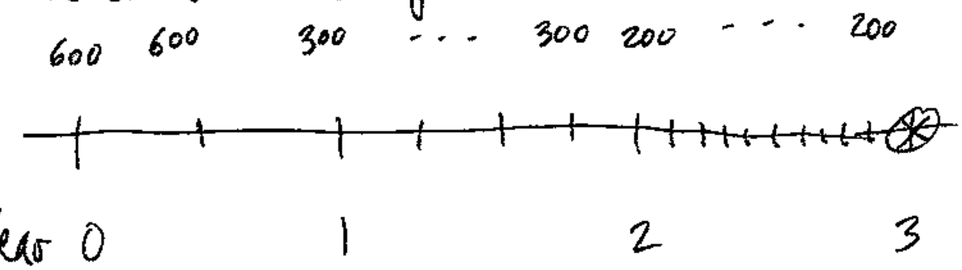
The question asks for $_{1.5}q_{60}$, which is the probability of death between age 60 and age 61.5:

$$\begin{aligned} {}_{1.5}q_{60} &= q_{60} + (1-q_{60})({}_{.5}q_{61}) \\ &= .02 + (1-.02)(.5)(.022) \\ &= .0308 \end{aligned}$$

B

2. This is a typical exam question on handling of nominal interest rates. You have three different nominal interest rates over a period of three years. You have varying frequency and amount of deposits, which also vary each year.

The first step is to write down all the cash flows on a time line diagram



The total payments each year are 1200. Now you need to determine the annual interest rate which is equivalent to the nominal interest given in the problem. Then you need to convert the interest compounding period to match the payment frequency

Calculations for Year	1	2	3
Rate given	$d^{(12)} = 6.0\%$	$i^{(3)} = 8.0\%$	$\delta = 7.0\%$
$1+i$ equals	$\left[1-\frac{d^{(12)}}{12}\right]^{-12}$	$\left[1+\frac{i^{(3)}}{3}\right]^{3}$	e^{δ}

2. continued

Calculations for Year	1	2	3
1+i (effective rate)	$(.995)^{-12}$	$(1.02667)^3$	$e^{.07}$
	$=1.0620$	$=1.08215$	$=1.0725$
Payment frequency	semi-annual	quarterly	semi-monthly
equivalent rate	$\left[1+\frac{i^{(2)}}{2}\right]^2$	$\left[1+\frac{i^{(4)}}{4}\right]^4$	$\left[1+\frac{i^{(6)}}{6}\right]^6$
	$\frac{i^{(2)}}{2}=(1.0620)^{.5}-1$	$\frac{i^{(4)}}{4}=(1.08215)^{1/4}-1$	$\frac{i^{(6)}}{6}=(1.0725)^{1/6}-1$
	$=3.0532\%$	$=1.9934\%$	$=1.1735\%$
Accumulated payments at end of year	$600\ddot{s}_{\overline{2}\rceil 3.05\%}$	$300\ddot{s}_{\overline{4}\rceil 1.99\%}$	$200\ddot{s}_{\overline{6}\rceil 1.17\%}$
	$=1255.52$	$=1261.01$	$=1250.26$

The question asks for the value of all the payments accumulated to the end of year 3. You need to adjust each year-end value above with the subsequent annual interest rates

$$X = 1255.52(1.08215)(1.0725) + 1261.01(1.0725) + 1250.26$$
$$= 4059.88$$

D

2012

3. This is a typical exam question on basic life contingencies. You must calculate the value of a certain and life annuity based on the formula given for l_x.

As usual, you should write down the payments on a time line diagram:

			1000	1000	1000	
Age	105	106	107	108	109	110

There are only three payments, since the value of l_{110} is zero using the given formula. The first two payments are the "certain part" of the annuity - but Smith must survive to age 106 to receive them:

$$X = v\left(\frac{l_{106}}{l_{105}}\right)1000\left[v + v^2 + v^3\left(\frac{l_{109}}{l_{106}}\right) + \text{zero}\right]$$

$$= \left(\frac{1-.2}{1-0}\right)\frac{1000}{1.07}\left[(1.07)^{-1} + (1.07)^{-2} + (1.07)^{-3}\left(\frac{1-.8}{1-.2}\right)\right]$$

$$= .8(934.58)\left[.9346 + .8734 + .8163(.25)\right]$$

$$= 1504.37$$

B

4. This is a typical question on actuarially equivalent benefits. Since the two annuities are actuarially equivalent, they have the same present values.

You are given various commutation functions which allow you to calculate the present value of the life annuity (option B):

$$\begin{aligned}
PV \text{ of } B &= 12000a_{40}^{(12)} \\
&= 12{,}000\left(\ddot{a}_{40}^{(12)}\right) - 1{,}000 \\
&= 12{,}000\left(\frac{N_{40}}{D_{40}} - \frac{11}{24}\right) - 1{,}000 \\
&= 12{,}000(12.9057) - 1{,}000 \\
&= 153{,}869
\end{aligned}$$

You can use the commutation functions to derive the value of the interest rate:

$$\frac{D_{x+1}}{D_x} = v p_x \Rightarrow \frac{607}{651} = \frac{p_{40}}{1+i}$$

$$\begin{aligned}
1+i &= (1 - q_{40})(651/607) \\
&= 1.07021
\end{aligned}$$

I am going to assume the actual interest rate is exactly 7.000%. This is based on the data used to calculate $1+i$. Since the D_x factors have 3 significant digits, the resulting $1+i$ only has 3 significant digits

(4) continued

Now you can calculate the present value of the monthly perpetuity (option A) using 7% interest.

$1+i = 1.07000$

With monthly payments, you need to derive the equivalent monthly interest rate j:

$(1+j)^{12} = 1.07000 \Rightarrow 1+j = (1.07)^{1/12} = 1.005654$

$$PV \text{ of } A = P\left[(1.005654)^{-1} + (1.005654)^{-2} + \dots\right]$$
$$= P/.005654$$
$$= 176.8614\,P$$

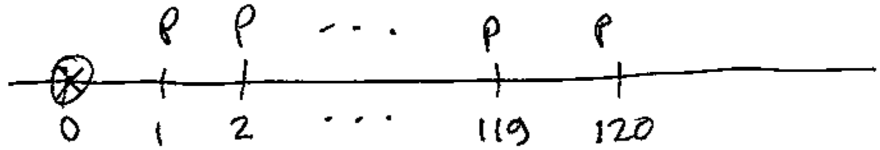
Now you can calculate the value of P, since the present values of both annuities are equal:

$$PV \text{ of } B = 153.869 = 176.8614\,P$$
$$P = 869.996$$

(B)

If you did not limit the interest rate to three significant digits, you would calculate the present value of annuity A with an annual interest rate of 7.021%. This gives a value for P of 872.51, which is in the same answer range.

5. This is a typical non-complicated loan question. You need to write down the loan payments on a time-line diagram. I'll use P for the original loan payment



The interest rate is given as 12% per year, compounded monthly. You can easily calculate the value of P, since the monthly interest rate is 1.0%:

$$12{,}000 = P\, a_{\overline{120}|1\%}$$
$$P = 12{,}000/69.70$$
$$= 172.17$$

To determine the value of X, you need to calculate the outstanding loan balance. After the first 48 loan payments, the balance is equal to the present value of the remaining 72 payments (120-48=72):

O/S balance at time 48 = $172.17\,(a_{\overline{72}|1\%})$

O/S balance at time 60 = $172.17\,(a_{\overline{72}|1\%})(1.01)^{12}$

$= 9{,}923$

Now you calculate the value of X for the last 60 payments

$$9{,}923 = X\, a_{\overline{60}|1\%}$$
$$X = 9923/44.9550 = 220.74$$

D

6 This is an unusual problem on annuity calculations. One key point is correctly interpreting the statement about 1% annual mortality improvements. My first guess on what that means was NOT correct

The first step in the solution is interpreting the symbol ${}_{20|2}a_{60}$. This is an immediate annuity for 2 years, deferred 20 years from age 60:

$$ {}_{20|2}a_{60} = \frac{D_{80}}{D_{60}} \left(a_{80:\overline{2|}} \right) $$

$$ = \frac{D_{80}}{D_{60}} \left(v\,p_{80} + v^2\,{}_2p_{80} \right) $$

Based on the information given, you can set up the calculation for X:

$$ X = (D_{80}/D_{60}) \left(\frac{p_{80}}{1.06} + \frac{p_{80}\,p_{81}}{(1.06)^2} \right) $$

$$ = \frac{D_{80}}{D_{60}} \left[\frac{1.06(.9521) + .9521(.9461)}{(1.06)^2} \right] $$

The correct interpretation of the mortality improvement is that the projected q_x after n years is given by $q_x(.99)^x$. At first, I tried adjusting the p_x value, which is simply wrong!

(6) continued

Now you must project the mortality improvements and set up the expression to calculate Y:

Age x	80	81
p_x	.9521	.9461
q_x	.0479	.0539
projection years from age 60	20	21
projected q_x	$.0479(.99)^{20}$ $=.0392$	$.0539(.99)^{21}$ $=.0436$
projected p_x	.9608	.9564

$$Y = (D_{80}/D_{60})\left[\frac{p_{80}}{1.06} + \frac{p_{80}p_{81}}{(1.06)^2}\right] \text{ using projected } q_x$$

$$= \frac{D_{80}}{D_{60}}\left[\frac{1.06\,p_{80} + p_{80}p_{81}}{(1.06)^2}\right]$$

$$\frac{Y}{X} = \frac{1.06(.9608) + .9608(.9564)}{1.06(.9521) + .9521(.9461)}$$

$$= \frac{1.9374}{1.9100}$$

$$= 1.01432$$

C

There is a slight variation in the solution, based on a different interpretation of the mortality improvement – see the next page.

(6) continued

Another way to calculate the qx values is to divide by $(1.01)^x$ instead of multiplying by $(.99)^x$. It gives almost the same numerical result

Age x	80	81
q_x	.0479	.0539
projected q_x	$.0479(1.01)^{-20}$	$.0539(1.01)^{-21}$
	$= .0393$	$= .0437$
projected p_x	.9607	.9563

$$\frac{Y}{X} = \frac{1.06(.9607) + .9607(.9563)}{1.06(.9521) + .9521(.9461)}$$

$$= \frac{1.9371}{1.9100}$$

$$= 1.01419$$

Here comes the tricky part – the answer ranges are shown to 5 significant digits, which is actually wrong. The values given for p_{80} and p_{81} only have 4 significant digits. As a result, any calculations using those values only has 4 significant digits.

This would produce an answer of 1.014, but that can't be used in this problem. You must round up to 1.0142, which barely (maybe) is still in range C.

7 This is a fairly typical exam question on life insurance values. The key to working the problem is writing down the insurance values at ages 50 and 53 in terms of l_x and d_x

$$A_{50} = vq_x + v^2\,{}_{1|}q_x + v^3\,{}_{2|}q_x + \dots$$
$$= \frac{vd_{50} + v^2 d_{51} + v^3 d_{52} + v^4 d_{53} + \dots}{l_{50}}$$

$$A_{53} = \frac{vd_{53} + v^2 d_{54} + \dots}{l_{53}}$$

You are given the net single premium for age 50. You can express the age 53 insurance based on the values at age 50

$$5{,}000 = 10{,}000\,A_{50}$$
$$A_{50} = .50$$

$$A_{53} = \left(A_{50} - \frac{vd_{50} - v^2 d_{51} - v^3 d_{52}}{l_{50}}\right)\frac{l_{50}}{l_{53}}(1.08)^3$$
$$= .50\frac{l_{50}(1.08)^3}{l_{53}} - \left(\frac{(1.08)^2 d_{50} + 1.08 d_{51} + d_{52}}{l_{53}}\right)$$

At each age, the d_x value is equal to 5:

$$A_{53} = .50\,(100/85)(1.08)^3 - [(1.08)^2(5) + 1.08(5) + 5]/85$$
$$= .5500 \qquad \Rightarrow 10{,}000\,A_{53} = 5500.42 \qquad C$$

(7) continued

The alternative solution is to use the relationship between insurances and annuities. This seems to be a bit longer, but also produces the same result

$A_x = 1 - d\ddot{a}_x \qquad A_{50} = 1 - d\ddot{a}_{50} \qquad A_{53} = 1 - d\ddot{a}_{53}$

$$\ddot{a}_{50} = 1 + v p_{50} + v^2\,{}_2p_{50} + v^3\,{}_3p_{50} + v^4\,{}_4p_{50} + \cdots$$
$$= 1 + v p_{50} + v^2 p_{50}(p_{51}) + v^3 p_{50}(p_{51})(p_{52})\ddot{a}_{53}$$

$$\ddot{a}_{53} = \frac{\ddot{a}_{50} - (1 + v p_{50} + v^2 p_{50} p_{51})}{v^3 p_{50} p_{51} p_{52}}$$
$$= \frac{\ddot{a}_{50} - [1 + (1.08)^{-1}(l_{51}/l_{50}) + (1.08)^{-2}(l_{52}/l_{50})]}{(1.08)^{-3}(l_{53}/l_{50})}$$

$10{,}000 A_{50} = 5{,}000$

$A_{50} = .50 = 1 - d\,\ddot{a}_{50}$

$\ddot{a}_{50} = .50/d$

$$\ddot{a}_{53} = \frac{(.50/d) - [1 + .95/1.08 + .90/(1.08)^2]}{.85/(1.08)^3}$$
$$= (6.750 - 2.651)/.6748$$
$$= 6.0744$$

$$10{,}000 A_{53} = 10{,}000\,(1 - d\,\ddot{a}_{53})$$
$$= 10{,}000\,(1 - 6.0744(.08/1.08))$$
$$= 5{,}500.42$$

C

8. This is a simplified problem on calculating the modified duration for a portfolio of 2 bonds. The key idea is knowing the definition of the modified duration. It is defined as the regular duration divided by $1+i$:

$$\text{modified duration} = \frac{\bar{d}}{1+i}$$

$$\text{regular duration} \quad \bar{d} = \left(\sum_{t=1}^{n} t v^t R_t\right) \Big/ \left(\sum_{t=1}^{n} v^t R_t\right)$$

The regular duration is defined as a weighted average of the time of payment for the series of payments. The weight for each payment is the present value of the payment.

You are told that the portfolio is a serial bond which is paid at two maturity dates. The easiest way to work the problem is to consider this as two bonds with face value of 5,000 which are redeemed at two dates.

The first step is to write down the coupons and redemption values on a time line diagram

Bond 1		250	250		250	250+5,000
Bond 2		250	250		250+5,000	
	0	1	2	...	9	10

(8) continued

Now you can write down the formula for the regular duration of the portfolio:

$$\bar{d} = \frac{1 \cdot 500v^1 + 2(500)v^2 + \ldots + 9(500)v^9 + 9(5000)v^9 + 10(5{,}250)v^{10}}{500\, a_{\overline{9}|.04} + 5{,}000v^9 + 5{,}250v^{10}}$$

The numerator can be rewritten using $Ia_{\overline{n}|}$

$$\bar{d} = \frac{500(Ia_{\overline{9}|.04}) + 45{,}000v^9 + 52{,}500v^{10}}{500(a_{\overline{9}|}) + 5{,}000v^9 + 5250v^{10}}$$

You must know the formula for $Ia_{\overline{n}|i}$. You could derive it, but that approach wastes time during the exam

$$Ia_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i} = \frac{(1+i)a_{\overline{n}|i} - nv^n}{i}$$

I use the HP-12C calculator, and I have it set up to only calculate immediate annuities on the EA-1 exam. That is why I rewrote the formula for $\ddot{a}_{\overline{n}|i}$ in the numerator of the fraction

$$Ia_{\overline{9}|.04} = \frac{1.04(a_{\overline{9}|.04}) - 9(1.04)^{-9}}{.04}$$

$$= (7.7327 - 6.3233)/.04$$

$$= 35.2366$$

(8) continued

NOTE – don't forget that you need to calculate the value of X, which is <u>not</u> the regular duration

$$X = \frac{\bar{d}}{1+i} = (1.04)^{-1}\left[\frac{500(Ia_{\overline{9}|i}) + 45{,}000(1.04)^{-9} + 52{,}500(1.04)^{-10}}{500\,a_{\overline{9}|.04} + 5000(1.04)^{-9} + 5250(1.04)^{-10}}\right]$$

$$= (1.04)^{-1}\left[\frac{500(35.2366) + 31{,}616 + 35{,}467}{3{,}718 + 3{,}513 + 3{,}547}\right]$$

$$= 7.5570$$

(B)

2012

9. This is a typical exam question on calculating the dollar weighted rate of return. The key to working these problems is carefully writing down all of the information on a time line diagram.

	1-1	4-1	7-1	10-1	11-1	12-31
Ben pmt		-20,000			-20,000	
MVA	2,000,000					2,200,000

Now you can write the formula for the dollar weighted return, based on simple interest for fractions of a year:

$$2{,}200{,}000 = 2{,}000{,}000(1+i) - 20{,}000\left(1+\tfrac{9}{12}(i)\right) - 20{,}000\left(1+\tfrac{2}{12}(i)\right)$$

$$240{,}000 = i\left[2{,}000{,}000 - \tfrac{9}{12}(20{,}000) - \tfrac{2}{12}(20{,}000)\right]$$

$$X = i = \frac{240{,}000}{1{,}981{,}667} = 12.11\%$$

(B)

As a quick check, you can use this value to calculate the expected market value at 12-31 under compound interest:

$$2{,}000{,}000(1+i) - 20{,}000(1+i)^{9/12} - 20{,}000(1+i)^{2/12} = 2{,}200{,}045$$

close enough!

10 This is a straightforward question on actuarially equivalent annuities. You are told that the present value of Jones annuity is 4 times that present value of Smith's annuity. This will allow you to solve for Smith's annual payment.

$$\text{PV of Smith (age 61)} = X(\ddot{a}_{61:\overline{5}|})$$

$$\text{PV of Jones (age 60)} = 20{,}000\left[\ddot{a}_{\overline{10}|.07} + \frac{D_{70}}{D_{60}}\ddot{a}_{70}\right]$$

One minor point of the problem is that you need to modify the formulas based on the present value factors given in the question.

$$\begin{aligned}\text{PV of Jones} &= 20{,}000\left[1.07(a_{\overline{10}|.07}) + \ddot{a}_{60} - \ddot{a}_{60:\overline{10}|}\right]\\ &= 20{,}000\left[1.07(7.0236) + 11.53496 - 7.26514\right]\\ &= 235{,}701\end{aligned}$$

$$\begin{aligned}\text{PV of Smith} &= X\left(1 + vp_{61}\,\ddot{a}_{62:\overline{4}|}\right)\\ &= X\left(1 + \frac{.99394}{1.07}(3.58056)\right)\\ &= 4.3260X = \text{PV of Jones' annuity}/4\\ &= 235{,}701/4\\ X &= 235{,}701/(4 * 4.3260)\\ &= 13{,}621\end{aligned}$$

C

11 This is a typical exam question on the expectation of life. The problem gives you e_x values, and you need to determine the probability of survival from age 75 to age 77.

$$X = 10{,}000\,({}_2p_{75})$$
$$= 10{,}000\,(p_{75})(p_{76})$$

$$e_x = p_x + {}_2p_x + {}_3p_x + \ldots$$
$$= p_x\,(1 + p_{x+1} + {}_2p_{x+1} + \ldots)$$
$$= p_x\,(1 + e_{x+1})$$

$$p_x = \frac{e_x}{1+e_{x+1}} \Rightarrow \quad p_{75} = \frac{e_{75}}{1+e_{76}} = \frac{10.5}{1+10.0} = .9545 \qquad p_{76} = \frac{e_{76}}{1+e_{77}} = \frac{10.0}{1+9.6} = .9434$$

$$X = 10{,}000\,(p_{75})(p_{76})$$
$$= 10{,}000\,(.9545)(.9434)$$
$$= 9{,}005$$

B

NOTE – If you limit the p_x values to 3 decimal places, you get a result of 9006 which is in the same answer range. The reason for doing this is that the first two e_x values only have 3 significant digits of data (and probably the third one too).

12 This is not really a question on actuarial equivalence, since no life contingencies are involved. You have two annuities with non-level payments, and the same present values.

Annuity A

	500	500	500	1000	⋯	1000	1000
0	1	2	3	4	⋯	11	12

For annuity A, you need to determine the monthly interest rate. You are given the nominal rate, compounded monthly, so this is $.08/12 = .6667\%$ per month.

$$PV \text{ of } A = 1000\, a_{\overline{12}|.6667\%} - 500\, a_{\overline{3}|.6667\%}$$
$$= 11{,}495.78 - 1{,}480.22$$
$$= 10{,}015.56$$

You could also calculate the present value as the sum of two annuity values. You could use this as a check of the prior calculation:

$$PV \text{ of } A = 500\, a_{\overline{3}|.6667\%} + v^3 (1000)\, a_{\overline{9}|.6667\%}$$

Annuity B

	X	X	2X	2X
0	1	2	3	4

(12) continued

For annuity B, you need to determine the quarterly interest rate that is equivalent to the nominal annual rate of 8%, compounded monthly:

$$(1+j)^4 = \left[1+\frac{i^{(12)}}{12}\right]^{12} \Rightarrow 1+j = \left(1+\frac{.08}{12}\right)^3$$
$$= 1.02013$$

You can write different expressions for the present value of annuity B. You can use the sum of annuities or the difference. I prefer the difference because it skips the step of discounting the value of the later annuity:

$$PV \text{ of } B = 10{,}015.56 = X\left(a_{\overline{2}|2.013\%} + 2v^2 a_{\overline{2}|2.013\%}\right)$$
$$= 2X\, a_{\overline{4}|2.013\%} - X\, a_{\overline{2}|2.013\%}$$
$$X = \frac{10{,}015.56}{2(3.8065)-1.9412}$$
$$= 1{,}765.85$$

A

13. This is an interesting variation from the typical exam questions on the joint and survivor annuity form of payment. For the first 20 years, you have a typical joint and survivor annuity. Start with the usual definition for 100% J+S:

Ignoring the 20 year "restriction":

$$100\%\text{ J+S annuity} = \ddot{a}_x + 100\%(\ddot{a}_y - \ddot{a}_{xy})$$

To reflect the 20 year restriction, add the 20 year payment term to each annuity:

$$\begin{array}{l}100\%\text{ J+S annuity}\\ \text{for first 20 years} = \ddot{a}_{x:\overline{20|}} + 100\%(\ddot{a}_{y:\overline{20|}} - \ddot{a}_{xy:\overline{20|}})\end{array}$$

After the first 20 years, you have a joint life annuity. Now you can write out the formula

$$PV = \ddot{a}_{x:\overline{20|}} + \ddot{a}_{y:\overline{20|}} - \ddot{a}_{xy:\overline{20|}} + (v^{20}\,{}_{20}p_{xy})\,\ddot{a}_{x+20:y+20}$$

As usual, you need to rewrite this to use the present value factors given in the problem

$$\begin{aligned}PV &= (\ddot{a}_x - {}_{20|}\ddot{a}_x) + (\ddot{a}_y - {}_{20|}\ddot{a}_y) - (\ddot{a}_{xy} - {}_{20|}\ddot{a}_{xy}) + {}_{20|}\ddot{a}_{xy}\\ &= 10.0 - 3.0 + 8.4 - 2.5 - (6.2 - 1.9) + 1.9\\ &= 10.5\end{aligned}$$

D

14 This is a simplified question on multiple decrement tables. The key idea is knowing the correct formula for multiple decrement probabilities when the single decrement table assumes uniform distribution of decrements (UDD).

In general, there are two ways to express the total probability of exit $q_x^{(\tau)}$. One formula uses the multiple decrement table probabilities, and the other uses single decrement table rates:

$$q_x^{(\tau)} = q_x^{(1)} + q_x^{(2)} + \cdots + q_x^{(n)} \quad \text{with } n \text{ decrement types}$$
$$= 1 - p_x'^{(1)} \cdot p_x'^{(2)} \cdots p_x'^{(n)}$$
$$= 1 - [1 - q_x'^{(1)}][1 - q_x'^{(2)}] \cdots [1 - q_x'^{(n)}]$$

With UDD in the single decrement tables, you have this relationship between a single decrement rate and the multiple decrement probabilities:

$$q_x'^{(n)} \doteq q_x^{(n)} / [1 - \tfrac{1}{2}(q_x^{(\tau)} - q_x^{(n)})]$$

$$q_x'^{(1)} \doteq q_x^{(1)} / [1 - \tfrac{1}{2}(q_x^{(2)})] \quad \text{with only 2 decrements}$$

$$q_x'^{(1)} [1 - \tfrac{1}{2} q_x^{(2)}] \doteq q_x^{(1)} = .03[1 - \tfrac{1}{2}(.10)]$$
$$= .0285$$

$$q_x^{(\tau)} = q_x^{(1)} + q_x^{(2)}$$
$$= .0285 + .10 = .1285$$

C

15. This is a typical exam question on asset values, realized gains and losses and unrealized gains and losses. A realized gain occurs when you sell an investment for more than book value. This realized gain affects both the market value and the book value of assets.

An unrealized gain is the excess of market value over the book value of assets. A reconciliation of book value of assets between two years includes any realized gain (or loss). A reconciliation of market value of assets between two years includes both realized G/L as well as the <u>change</u> in the unrealized G/L.

This problem gives you book and market values at 12-31-2011 and 12-31-2012. You have all the cash flows and realized and unrealized G/L. You can construct the reconciliation of book and market values to determine the missing asset values:

12/31/2012
Book value Y = 5,000,000 + cash flows + realized gain/loss

12/31/2012
Market value 5,335,000 = X + cash flows + realized gain/loss + Δ(unrealized gain/loss)

(15) continued

2012 cash flows = 600,000 + 315,000 - 250,000 - 60,000
= 605,000

12/31/2012
Book Value Y = 5,000,000 + 605,000 + 465,000 RG
= 6,070,000

12/31/2012
market value 5,335,000 = X + 605,000 + 465,000 - 535,000 UG
= X + 535,000
X = 4,800,000

X - Y = 4,800,000 - 6,070,000
= -1,270,000

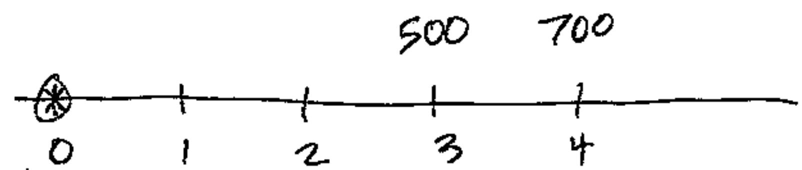
A

2012

16 This is a typical exam question on yield curves, spot rates and forward rates. If you understand the definitions of those items, the question is straightforward.

The spot rates reflect the yield to maturity for the stated time period. The forward interest rate is the return for a single year.

The next step is to write down the cash flows on a time line diagram:



The problem gives you the spot rate for the first two years. You must combine that with the forward rate for the 3rd year to calculate the present value of the 500 payment at time 3.

$$950 = 500(1+x)^{-1}(1.05)^{-2} + 700(1.07)^{-4}$$

$$500(1+x)^{-1}(1.05)^{-2} = 950 - 700(1.07)^{-4}$$

$$\frac{500(1.05)^{-2}}{950-700(1.07)^{-4}} = 1+x$$

$$1+X = 1.09025$$

$$X = 9.025\%$$

D

17 This is a typical question on the relationship between insurance and annuities. The key is knowing various identity formulas. The first step in the problem is solving for the interest rate

$$A_x = 1 - d\ddot{a}_x$$
$$d\ddot{a}_x = 1 - A_x$$
$$\ddot{a}_x = \frac{1-A_x}{d} = .82/d$$
$$\ddot{a}_{x+1} = \frac{(1-A_{x+1})}{d} = .81/d$$
$$\ddot{a}_{x+2} = \frac{1-A_{x+2}}{d} = .80/d$$

$$v\,p_x\,\ddot{a}_{x+1} = a_x$$
$$\frac{p_x\ddot{a}_{x+1}}{a_x} = 1+i$$
$$1+i = p_x\,\ddot{a}_{x+1} / (\ddot{a}_x - 1.0)$$
$$= p_{x+1}\ddot{a}_{x+2} / [\ddot{a}_{x+1} - 1.0]$$

$$1+i = \frac{(1-q_{x+1})\,\ddot{a}_{x+2}}{\ddot{a}_{x+1} - 1.0}$$
$$= \frac{.98875(.80/d)}{(.81/d) - 1.0}$$
$$1+i = \frac{.98875(.80)}{.81 - d}$$
$$v = \frac{.81 - (1-v)}{.98875(.80)}$$
$$v(.98875)(.80) = .19 + v$$
$$.98875(.80) = .19(1+i) + 1$$
$$1+i = 1.10$$

(17) continued

That took quite a few manipulations, but the solution is almost finished:

$$v p_x \ddot{a}_{x+1} = a_x$$

$$p_x = \frac{(\ddot{a}_x - 1.0)(1+i)}{\ddot{a}_{x+1}}$$

$$= \frac{(.82/d - 1.0)(1+i)}{.81/d}$$

$$= \frac{(.82 - d)(1.10)}{.81}$$

$$= \left[\frac{.82 - (1-v)}{.81}\right] 1.10$$

$$= \frac{1.10(.82) - 1.10 + 1}{.81}$$

$$p_x = .9901$$

$$1000 q_x = 9.8765$$

A

There is an alternate solution technique that does not involve using $A_x = 1 - d\ddot{a}_x$

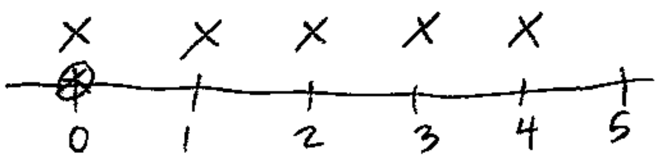
Instead, you can start with

$$A_x = v q_x + v^2 p_x q_{x+1} + v^3 {}_2p_x q_{x+2} + \ldots$$

and use the relationship between A_x and A_{x+1}:

$$A_x = v q_x + v p_x A_{x+1} \Rightarrow q_x = \frac{(1+i) A_x - A_{x+1}}{1 - A_{x+1}}$$

18 This is a typical exam question on loans. Repayment plan #1 involves annual payments for 5 years. You need to determine the equivalent annual interest rate first - then you can calculate the loan payment X:



$$75{,}000 = X\,\ddot{a}_{\overline{5}|i}$$

$$1+i = \left[1 - \frac{d^{(4)}}{4}\right]^{-4}$$

$$= \left[1 - \frac{.076225}{4}\right]^{-4}$$

$$= 1.080$$

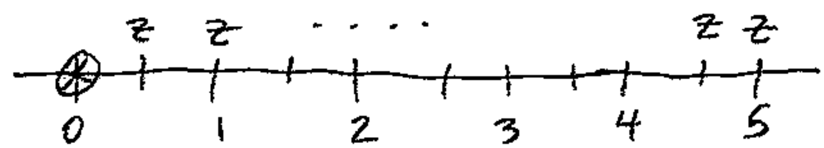
$$\therefore\ X = 75{,}000 \,/\, \left[1.08\, a_{\overline{5}|.08}\right]$$

$$= 17{,}393$$

A side note - since I use the HP-12C calculator, it is set up to always calculate immediate annuity values. That avoids silly errors caused by jumping back and forth between calculating annuities due versus annuities immediate.

(18) continued

Now you should set up Repayment plan #2 on a time line diagram. This uses payments at the end of each six month period:



$$75{,}000 = Z\,(a_{\overline{10}|j})$$

You need to determine the semi-annual rate of interest, which matches the payment period of six months:

$$(1+j)^2 = \left(1-\frac{d^{(4)}}{4}\right)^{-4} = 1.08 \Rightarrow 1+j = (1.08)^{1/2} = 1.03923$$

$$Z = \frac{75{,}000}{a_{\overline{10}|3.923\%}} = 9{,}211$$

$$Y = 2(9{,}211) = 18{,}423$$

$$|X-Y| = |17{,}393 - 18{,}423| = 1{,}030$$

C

19 This is a typical exam question on present value of death benefits. Instead of a simple lump sum, this death benefit is a series of payments.

The first step is to write down the formula for the present value of the annuity and the death benefit. Then you need to adjust the formula to use the present value factors given:

$$P = 1000\left(v^{20}\,{}_{20}p_{45}\,\ddot{a}_{65}\right) + \left(\ddot{a}_{\overline{5}|7\%}\right)1000\,A^{1}_{45:\overline{20}|}$$

The annuity payments don't start for 20 years. In the next 20 years, the death benefit starts at the end of the year of death. You may need to think carefully to get the correct expression for the death benefit - it is a bit confusing to use an annuity due.

$$P = 1000\,\ddot{a}_{65}(D_{65}/D_{45}) + 1000\left(\ddot{a}_{\overline{5}|.07}\right)\left(A_{45:\overline{20}|} - D_{65}/D_{45}\right)$$

$$= 1000\left[\left(\ddot{a}_{45} - \ddot{a}_{45:\overline{20}|}\right) + \ddot{a}_{\overline{5}|.07}\left(1 - d\,\ddot{a}_{45:\overline{20}|} - v^{20}\,{}_{20}p_{45}\right)\right]$$

Now you can use the factors given in the problem:

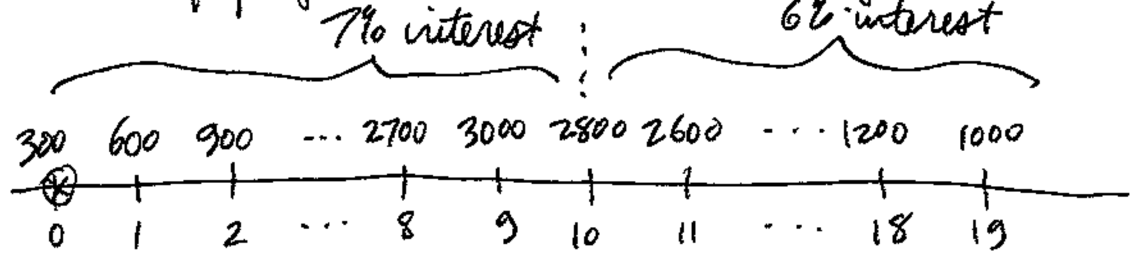
$$P = 1000\left[13.1949 - 10.9961 + 1.07(4.1002)\left(1 - \frac{.07}{1.07}(10.9961) - \frac{.8771}{(1.07)^{20}}\right)\right]$$

$$= 1000\left[2.1988 + .2369\right]$$

$$= 2436$$

(b)

20. This is a typical question on increasing/decreasing annuities. The first step is writing down the series of payments on a time-line diagram



There are 10 payments that increase by 300, followed by 10 payments that decrease by 200. There are slightly different ways of writing this series of payments' present value. You can have either the first 9 or 10 payments as part of the increasing annuity, and the rest as part of the decreasing annuity.

$$X = 300\, I\ddot{a}_{\overline{10}|.07} + (1.07)^{-10}\left[3000\, \ddot{a}_{\overline{10}|.06} - 200\, I\ddot{a}_{\overline{10}|.06}\right]$$

One definition you should memorize is the formula for $I\ddot{a}_{\overline{n}|i} = (1+i)\left(\dfrac{\ddot{a}_{\overline{n}|i} - nv^n}{i}\right)$

$$X = 300(1.07)\left[\frac{\ddot{a}_{\overline{10}|.07} - 10v^{10}}{.07}\right] + (1.07)^{-10}\left[3000(1.06)a_{\overline{10}|.06} - 200(1.06)\left[\frac{\ddot{a}_{\overline{10}|.06} - 10v^{10}}{.06}\right]\right]$$

$$= 321\left[\frac{1.07(a_{\overline{10}|.07}) - 10(1.07)^{-10}}{.07}\right] + (1.07)^{-10}\left[3180\, a_{\overline{10}|.06} - 212\left[\frac{1.06(a_{\overline{10}|.06}) - 10(1.06)^{-10}}{.06}\right]\right]$$

$$= 11{,}151.26 + 7{,}914.51$$

D

$$= 19{,}065.77$$

2) This is a simplified exam question on calculations involving insurance values. The key is knowing the relationship between insurance and annuities:

$$A_x = 1 - d\ddot{a}_x$$

$$X = 1000A_{60}$$
$$= 1000(1 - d\ddot{a}_{60})$$

You also need to derive the value of the life annuity at age 61:

$$v p_{60} \ddot{a}_{61} = a_{60}$$
$$v(1-q_{60})(1+a_{61}) = \ddot{a}_{60} - 1.0$$

$$\ddot{a}_{60} = 1 + (1.06)^{-1}(1-.004803)(1+11.5069)$$
$$= 12.7423$$

$$X = 1000(1 - d\ddot{a}_{60})$$
$$= 1000\left(1 - \frac{.06}{1.06}(12.7423)\right)$$
$$= 278.74$$

B

22. This is a typical exam question on calculating the probability of survival. You must know what the S(x) function is — it is the probability of survival from age zero to age x:

$$S(x) = {}_xp_0$$

The problem asks for the value of ${}_{30|10}q_{10}$, so must also interpret this symbol. It is a 30 year deferred probability of death over a 10 year period for someone who is currently age 10:

$${}_{30|10}q_{10} = ({}_{30}p_{10})({}_{10}q_{40})$$

One minor trick to the problem is that the S(x) function is only valid up to age 25. You must use the formula for l_x beyond age 25.

You need to split up the calculations for ${}_{30|10}q_{10}$ to separately handle ages 10 to 25, and then ages 25 to 50:

$${}_{30|10}q_{10} = ({}_{15}p_{10})({}_{15}p_{25})(1 - {}_{10}p_{40})$$

$${}_{15}p_{10} = \frac{{}_{25}p_0}{{}_{10}p_0} = \frac{S(25)}{S(10)} = \frac{1 - \frac{25}{250}}{1 - \frac{10}{250}} = \frac{.90}{.96}$$

$$= .9375$$

(22) continued

$$ {}_{15}p_{25}(1 - {}_{10}p_{40}) = \frac{l_{40}}{l_{25}}\left(1 - \frac{l_{50}}{l_{40}}\right) $$

$$ = \frac{l_{40} - l_{50}}{l_{25}} $$

$$ = \frac{1000(100-40) - 1000(100-50)}{1000(100-25)} $$

$$ = \frac{60-50}{75} $$

$$ = .1333 $$

$$ {}_{30|10}q_{10} = {}_{15}p_{10}({}_{15}p_{25})(1 - {}_{10}p_{40}) $$

$$ = .9375(.1333) $$

$$ = .1250 $$

B

23 This is a typical exam question on callable bonds. You need to set up the bond price for each of the four different call dates, allowing for the different redemption values. It is a bit sneaky that the redemption at the end of the 20th year is not 1,000!

Once you calculate the price for each of the four different call dates, the final answer is the lowest value. This is the only price that would guarantee the purchaser earns 6.0% regardless of the call date. It does not matter which formula you use to calculate the bond price. The simplest choice is $P = Fr(a_{\overline{n}|i}) + Cv^n$

$$P_{17} = 1000(.05)a_{\overline{17}|.06} + 1{,}050(1.06)^{-17}$$
$$= 523.86 + 389.93$$
$$= 913.80$$

$$P_{18} = 1000(.05)a_{\overline{18}|.06} + 1{,}025(1.06)^{-18}$$
$$= 541.38 + 359.10$$
$$= 900.48$$

$$P_{19} = 1000(.05)a_{\overline{19}|.06} + 1{,}010(1.06)^{-19}$$
$$= 557.91 + 333.82$$
$$= 891.72$$

$$P_{20} = 1000(.05)a_{\overline{20}|.06} + 1050(1.06)^{-20}$$
$$= 573.50 + 327.39$$
$$= 900.89$$

The lowest of the four values is 891.72 B

24 This is a relatively straightforward question on multiple decrement tables. You are given information on four different decrement types, and you are told that they conform to uniform distribution of decrements from age 40 to 41.

You need to determine $q'^{(d)}_{40}$, which is the rate of mortality for the single decrement table. There are some formulas which give this value on an approximate basis. The usual formula for a two decrement table is

$$q'^{(1)}_{x} \doteq q^{(1)}_{x} / [1 - \tfrac{1}{2} q^{(2)}_{x}]$$

It is easy to extend this to more than two decrements, since the denominator is based on expected exits due to the other decrements:

$$q'^{(1)}_{x} \doteq q^{(1)}_{x} / [1 - \tfrac{1}{2} q^{(2)}_{x} - \tfrac{1}{2} q^{(3)}_{x} - \tfrac{1}{2} q^{(4)}_{x}] \quad \text{where } x = 40$$

$$q'^{(d)}_{x} \doteq q^{(d)}_{x} / [1 - \tfrac{1}{2} q^{(2)}_{x} - \tfrac{1}{2} q^{(3)}_{x} - \tfrac{1}{2} q^{(4)}_{x}]$$

$$= \frac{12/1500}{1 - \frac{1}{2}\left[\frac{30}{1500} + \frac{20}{1500} + \frac{10}{1500}\right]}$$

$$= \frac{12}{1500 - \frac{1}{2}[30 + 20 + 10]}$$

$$= .008163$$

B

(24) continued

There is a different formula you can use, which gives slightly different results:

$${}_tp_x^{\prime(1)} = \left[{}_tp_x^{(T)}\right]^{q_x^{(1)}/q_x^{(T)}}$$

This gives non-estimated values, and is applicable when the multiple decrement table is based on either constant force of decrement or uniform distribution of decrements.

$${}_tp_x^{\prime(d)} = 1 - {}_tq_x^{\prime(d)} = \left[{}_tp_x^{(T)}\right]^{q_x^{(d)}/q_x^{(T)}}$$

$q_x^{(d)} = 12/1500$ where $x = 40$

$$q_x^{(T)} = (30+20+10+12)/1500$$
$$= 72/1500$$

$$p_{40}^{\prime(d)} = \left[p_{40}^{(T)}\right]^{12/72}$$
$$= [1 - 72/1500]^{1/6}$$
$$= .991835$$
$$q_{40}^{\prime(d)} = .008165$$

B

25. This idea of calculating the weighted average assumed retirement age has appeared on prior exams - see 2011 #31. This time, the problem is quite complicated, and the arithmetic is simply ridiculous.

The first thing you need to do is identify what decrements apply at each age. Then you can calculate a "simple" weighted average retirement age. There are special decrements that apply at the first age the participant is eligible for unreduced retirement (age 60 with 15 years) and at normal retirement (age 62 with 10 years or age 65 with 5 years).

t	Age $58+t$	service $13+t$	$q_{x+t}^{(r)}$	"special"
0	58	13	.05	
1	59	14	.05	
2	60	15	.15	YES
3	61	16	.10	
4	62	17	.40	YES
5	63	18	.25	
6	64	19	.25	
7	65	20	.25	
8	66	21	1.00	

(25) continued

Now you need to calculate the probability that the participant survives to each future age, and the weight to use for the average retirement age

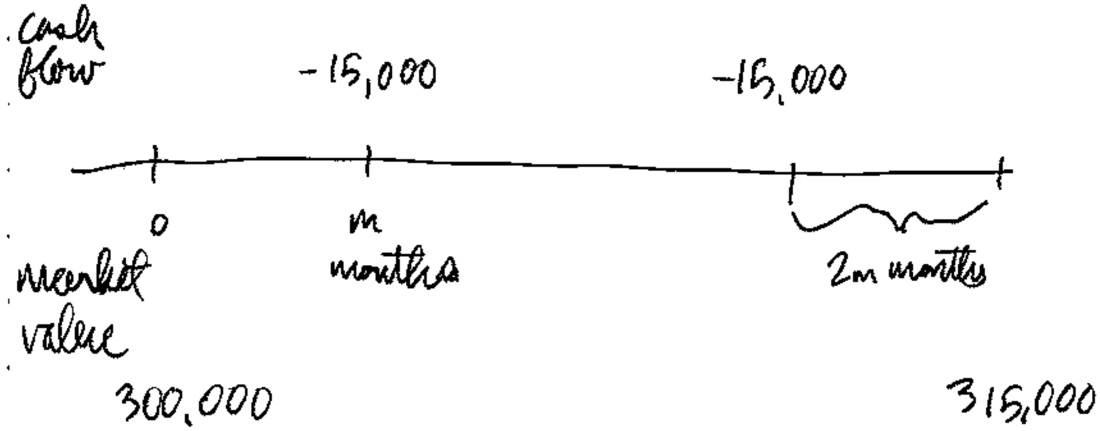
	(1)	(2)	(3)	(4)	(1)(2)(3)
	Age			$=1-(3)$	weighted
t	$58+t$	${}_tp_{58}^{(T)}$	$q_{58+t}^{(r)}$	$p_{58+t}^{(T)}$	calculation
0	58	1.00	.05	.95	2.9000
1	59	.9500	.05	.95	2.8025
2	60	.9025	.15	.85	8.1225
3	61	.7671	.10	.90	4.6795
4	62	.6904	.40	.60	17.1222
5	63	.4142	.25	.75	6.5244
6	64	.3107	.25	.75	4.9710
7	65	.2330	.25	.75	3.7865
8	66	.1748	1.00	—	11.5342
					62.44

C

The values in column 2 are calculated by multiplying the prior year's column 2 value by the prior year's column 4 value. This is the probability of not retiring at any prior age, and surviving to age x+t.

26 This is a typical question on calculation of the rate of return on assets. You are given several cash flows and the dollar weighted rate of return. You need to solve for the unknown m months time period.

You should write down the information on a time line diagram. You do not need to calculate the market values before and after each cash flow, since you do NOT need to calculate the time-weighted rate of return.



Now you should carefully write down the formula for the dollar weighted return - it is a bit confusing

$$300{,}000(1+i) - 15{,}000\left[1 + \left(\frac{12-m}{12}\right)i\right] - 15{,}000\left[1 + \left(\frac{2m}{12}\right)i\right] = 315{,}000$$

$$300{,}000(1.16) - 15{,}000\left[2 + \left(\frac{12+m}{12}\right).16\right] = 315{,}000$$

$$2 + \left(1 + \frac{m}{12}\right).16 = \frac{348{,}000 - 315{,}000}{15{,}000} \Rightarrow m = 3.0 \qquad D$$

2012

27 This is a messy / typical problem on annuity calculations with multiple lives. It is made more confusing by having all three participants with the same age.

There are two keys to working this problem. One is knowing how to use reversionary annuities to directly write down the present values based on exactly N participants being alive.

The second key is to write down general formulas based on ages w, y and z instead of using the same age for all three. This is less confusing, and makes it easier to be sure you have written down the correct annuity formulas:

$$X = 1000\, a_{yzw} + 500(a_y - a_{y:\overline{zw}}) + 750(a_{yz} - a_{yzw}) + 500(a_z - a_{z:\overline{yw}}) + 750(a_{yw} - a_{yzw}) + 500(a_w - a_{w:\overline{yz}}) + 750(a_{wz} - a_{yzw})$$

Since there are three participants, there are three cases where exactly one annuitant is alive. The reversionary annuity is the difference between two values. One is the life annuity for the individual, and the other is a joint and last survivor annuity payable while that individual is alive, and at least one of the other two is alive.

(27) continued

Now you need to expand the terms for the three reversionary annuities for \$500:

$$a_{\overline{zw}} = a_z + a_w - a_{wz}$$
$$a_y - a_{y:\overline{zw}} = a_y - (a_{yz} + a_{yw} - a_{ywz})$$
$$a_z - a_{z:\overline{yw}} = a_z - (a_{zy} + a_{zw} - a_{ywz})$$
$$a_w - a_{w:\overline{yz}} = a_w - (a_{wy} + a_{wz} - a_{ywz})$$

Now you can combine all the various terms

$$\Sigma = a_y + a_z + a_w - 2a_{yz} - 2a_{yw} - 2a_{zw} + 3a_{yzw}$$

The last step is writing out the annuity terms for X, based on all three participants at age 64

$$X = 1000\,a_{64:64:64} + 500(3)(a_{64} - 2a_{64:64} + a_{64:64:64}) + 750(3)(a_{64:64} - a_{64:64:64})$$

$$= 1000(10.0) + 500(3)[25 - 2(15) + 10] + 750(3)(15 - 10)$$
$$= 28{,}750$$

C

28 This is a typical exam question on calculations involving joint and survivor annuities. You have three different annuities that are actuarially equivalent, so the present values of all three are the same.

Since no annuity factors are given, this will be an algebra problem. The first step is to set up formulas for each of the payment forms:

Option 1: $1000\, a_z$ — assume Smith is age z

Option 2: $X\,[a_z + 50\%\,(a_y - a_{yz})]$ — Smith's spouse is age y

Option 3: $875\,[a_z + 100\%\,(a_y - a_{yz})]$

At first blush, this can't be solved. You will have three simultaneous equations in four unknowns. But there is a standard trick you can use

$$1000\, a_z = X\,[a_z + 50\%\,(a_y - a_{yz})]$$

$$1000\, a_z = 875\,[a_z + 100\%\,(a_y - a_{yz})]$$

Next, divide both sides of the equation by a_z.

(28) continued

$$1000 = X\left[1 + 50\%\left(\frac{a_y - a_{yz}}{a_z}\right)\right]$$

$$1000 = 875\left[1 + 100\%\left(\frac{a_y - a_{yz}}{a_z}\right)\right]$$

If you look carefully, you don't really need to treat a_y, a_{yz} and a_z as three separate variables. Replace the ratio of the reversionary annuity to the life annuity with a new variable R:

$$1000 = X\left[1 + .5R\right]$$

$$1000 = 875\left[1 + R\right]$$

Now you have two equations in two unknowns. You can calculate the value of R from the second equation, and then solve for the value of X:

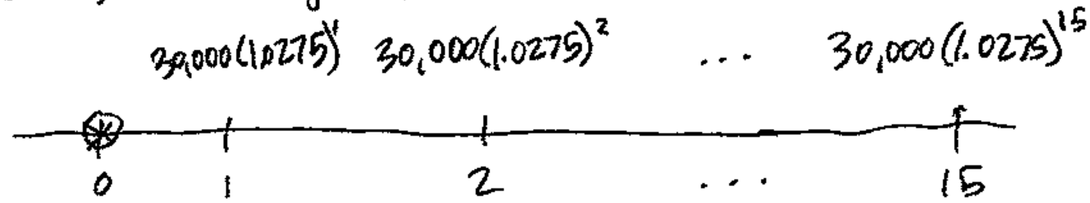
$$R = \frac{1000}{875} - 1$$
$$= .1429$$

$$X = \frac{1000}{1 + .5(.1429)}$$
$$= 933.33$$

D

2012

29. It sounds complicated, but this is really a question on increasing annuity calculations. The key point is writing down the payments on a time-line diagram.



I did not write down the initial payment of 30,000 since the question asks for the present value of the subsequent payments

$$X = 30{,}000\left[\frac{1.0275}{1.06} + \dots + \left(\frac{1.0275}{1.06}\right)^{15}\right]$$

$$= 30{,}000\, a_{\overline{15}|j} \quad \text{where } 1+j = 1.06/1.0275$$

$$= 1.03163$$

$$= 353{,}951$$

B

30 This is a typical exam question on select and ultimate decrements. The key idea is that you go through the table from left to right until you leave the select period rates. Then you go down the column of the ultimate mortality rates on the right side.

$${}_6p_{[60]+1} = p_{[60]+1} \cdot p_{[60]+2} \cdot p_{63} \cdot p_{64}\, p_{65}\, p_{66}$$

$$= (1-.012038)(1-.013454)(1-.014162)(1-.015509)(1-.017010) \times (1-.018685)$$

$$= .91250$$

B