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SPRING 2013 EA-1
EXAM SOLUTIONS

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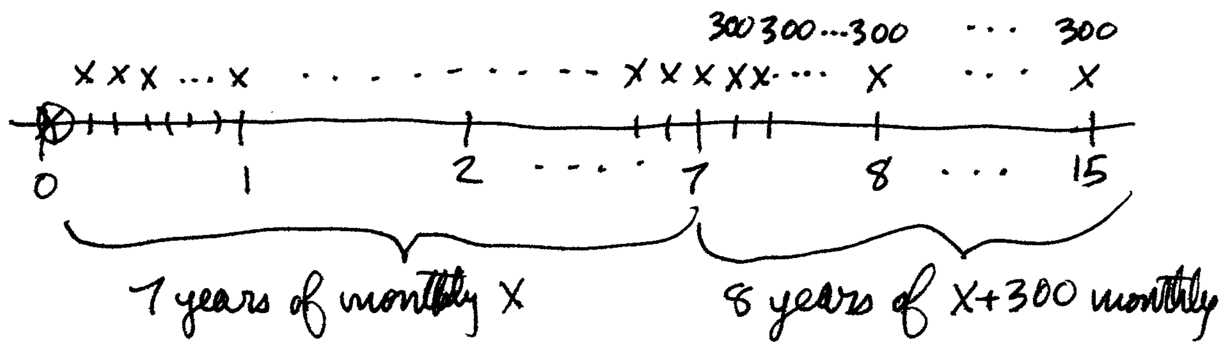
1 This is a typical exam question on the handling of nominal versus effective interest rates. You are given a series of monthly payments. You must convert the annual effective rate of 8% to an equivalent monthly rate:

$$(1+j)^{12} = 1.08$$
$$1+j = (1.08)^{1/12}$$
$$j = .6434\%$$

You should write down the series of payments on a time line diagram. You have monthly payments of X for 15 years, plus 8 years of monthly payments of \$300



$$20{,}600 = X a_{\overline{180}|j} + (1.08)^{-7}(300) a_{\overline{96}|j}$$

The key to working this problem is counting the numbers of payments correctly.

$$X = [20{,}600 - (1.08)^{-7}(300) a_{\overline{96}|j}] / a_{\overline{180}|j}$$
$$= [20{,}600 - 12{,}508] / 106.4276$$
$$= 76.04$$

C

2 This is a typical exam question involving joint and survivor annuities. One thing that can make this easier to solve is knowledge of reversionary annuities.

You are told the present value of annuity A is 116. You can use this to solve for the value of R, and then you can calculate the value of X.

$$116 = R\ddot{a}_{xy} + 2R(\ddot{a}_x - \ddot{a}_{xy}) + 2R(\ddot{a}_y - \ddot{a}_{xy})$$

The two terms with the difference between a single life annuity and the joint life annuity are reversionary annuities. This only pays 2R to the single life after the death of the other annuitant.

$$116 = R(7) + 2R(10-7) + 2R(15-7)$$
$$= 7R + 6R + 16R$$
$$R = 116/29 = 4.0$$

One minor trick to the question is that annuity B is not an annuity due - so you must be careful to write the correct expression:

$$X = R\,a_{xy} + \tfrac{1}{2}R(a_x - a_{xy}) + \tfrac{1}{2}R(a_y - a_{xy})$$

(2) continued

Now you can rewrite this expression using annuity due factors. Note that $a_x - a_{xy}$ has the same value as $\ddot{a}_x - \ddot{a}_{xy}$.

$$X = R(\ddot{a}_{xy} - 1.0) + \tfrac{1}{2}R(\ddot{a}_x - \ddot{a}_{xy}) + \tfrac{1}{2}R(\ddot{a}_y - \ddot{a}_{xy})$$
$$= R(7-1) + .5R(10-7) + .5R(15-7)$$
$$= 6R + 1.5R + 4R$$
$$= 11.5(4)$$
$$= 46.0$$

E

3 This is a typical exam question on present value calculations with commutation functions. You must value a monthly life annuity due at age 65, plus a ten year term insurance. You need to know the relationship between annuities and insurance:

$$A_x = 1 - d\ddot{a}_x$$

$$d = iv$$

$$A_x = 1 - \frac{.07}{1.07}\left(\frac{N_x}{D_x}\right)$$

$$A^1_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{D_x}$$

$$= A_x - \frac{D_{x+n}}{D_x} A_{x+n}$$

$$A^1_{65:\overline{10}|} = A_{65} - \frac{D_{75}}{D_{65}} A_{75}$$

$$= 1 - d\ddot{a}_{65} - \frac{D_{75}}{D_{65}}(1 - d\ddot{a}_{75})$$

$$= 1 - \frac{.07}{1.07}\left(\frac{8872}{965}\right) - \frac{346}{965}\left(1 - \frac{.07}{1.07}\left(\frac{2{,}379}{346}\right)\right)$$

$$= 1 - .06542(9.1938) - .3585(1 - .06542(6.8757))$$

$$= 1 - .6015 - .1973$$

$$= .2013$$

$$X = 600\,\ddot{a}^{(12)}_{65} + 10{,}000\left(A^1_{65:\overline{10}|}\right)$$

$$\ddot{a}^{(12)}_x \doteq \ddot{a}_x - \frac{11}{24} \Rightarrow \qquad \ddot{a}^{(12)}_{65} \doteq 9.1938 - .4583 = 8.7354$$

(3) continued

$$X = 600(8.7354) + 10,000(.2013)$$
$$= 5241 + 2013$$
$$= 7,254$$

(B)

2013

4 This is a typical question on calculation of fund balances with varying rates of interest. The key to the problem is knowing how to convert between annual and effective rates of interest.

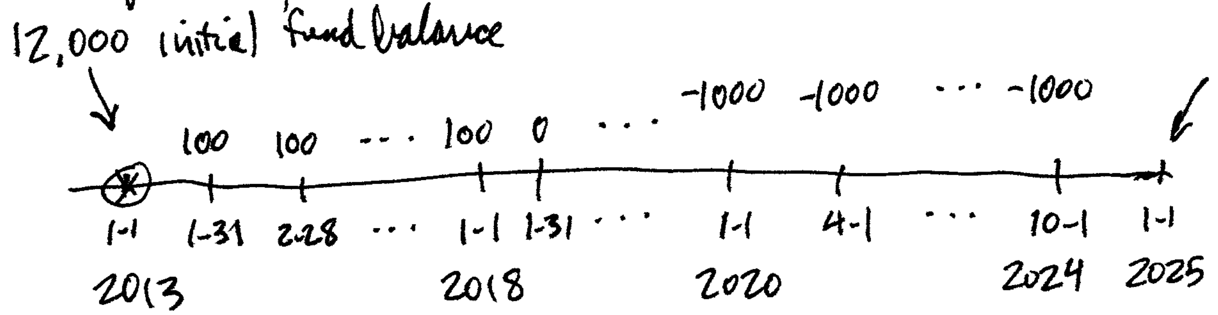
The monthly rate of interest is .6667% = .08/12. For handling the quarterly withdrawals, you need to convert the monthly rate to an equivalent quarterly interest rate.

Annual rate k: $1+k = (1.006667)^{12}$

Quarterly rate j: $1+j = (1.006667)^{3}$

$j = 2.01\%$

You should write all the cash flows on a time line diagram. The monthly deposits are on the last day of the month, and the quarterly withdrawals are on the first day of the quarter:



$$X = (1.006667)^{12(12)}\left[12{,}000 + 100\, a_{\overline{60}|.6667}\right] - 1000\, \ddot{s}_{\overline{20}|.0201}$$

The present value of the 12,000 balance and the 60 monthly deposits is brought forward to 1-1-2025. Then the 20 quarterly withdrawals are accumulated to the same date.

(4) continued

The calculation of annuities on my HP-12C is always done using immediate annuities - to avoid silly errors that happen when I forgot if I used annuity due versus annuity immediate in the past: $\ddot{s}_{\overline{20}|j} = (1+j)s_{\overline{20}|j}$

$$X = 2.6034(12{,}000 + 100(49.3184)) - 1000(1.0201)(24.3297)$$
$$= 44{,}080 - 24{,}820$$
$$= 19{,}261$$

E

5 This is a typical exam question on calculation of duration, specifically the modified duration. Duration is a calculation of the average weighted time of payment for a series of payments.

modified duration $X = \dfrac{\bar{d}}{1+i}$

regular Duration (or macaulay duration) $\bar{d} = \dfrac{\sum t\, P_t v^t}{\sum P_t v^t}$

The weight for each payment is the present value of each payment. Let the mortgage payment be P - you don't need to calculate the value, since it cancels out of both the numerator and the denominator

$$\bar{d} = \frac{P[1v^1 + 2v^2 + 3v^3 + \ldots + 20v^{20}]}{P[v^1 + v^2 + v^3 + \ldots + v^{20}]}$$

$$= \frac{Ia_{\overline{20}|.04}}{a_{\overline{20}|.04}}$$

You should memorize the formulas for $Ia_{\overline{n}|i}$ and $Da_{\overline{n}|i}$, since they will save you time when working exam problems.

$$Ia_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

$$= a_{\overline{n}|} + \frac{a_{\overline{n}|} - nv^n}{i}$$

(5) continued

$$\bar{d} = \frac{a_{\overline{20}|.04} + \dfrac{a_{\overline{20}|.04} - 20v^{20}}{.04}}{a_{\overline{20}|.04}}$$

$$= 1 + \frac{1}{.04} - \left[\frac{20v^{20}}{i\,a_{\overline{20}|.04}}\right]$$

$$= 1 + 25 - \frac{20(1.04)^{-20}}{1-(1.04)^{-20}}$$

$$= 9.2091$$

$$X = \frac{\bar{d}}{1+i}$$

$$= 9.2091/1.04$$

$$= 8.8549$$

(A)

One silly mistake is to calculate the regular duration and forget that the question asks for the modified duration. That gives the incorrect answer range of B.

6 This is a typical question on select and ultimate mortality rates. Smith is currently age 21, and was insured at age 20. The select period ends after 3 years, when Smith reaches age 23.

The problem asks for the probability that Smith dies between age 23 and age 24:

$$X = {}_2p_{[20]+1} - {}_3p_{[20]+1}$$
$$= (p_{[20]+1})(p_{[20]+2})\left[1 - p_{20+3}\right]$$
$$= (1-q_{[20]+1})(1-q_{[20]+2})\left[1-(l_{24}/l_{23})\right]$$

Now you can calculate the values of the select mortality rates, based on the formula given:

$$q_{[x]} = (1/2)q_x \qquad q_{[x]+1} = (2/3)q_{x+1} \qquad q_{[x]+2} = (3/4)q_{x+2}$$

$$p_x = \frac{30-x}{31-x} \qquad q_x = \frac{1}{31-x}$$

$$p_{23} = 7/8$$
$$q_{[20]+1} = (2/3)q_{21} = (2/3)(1/10) = 2/30$$
$$q_{[20]+2} = (3/4)q_{22} = (3/4)(1/9) = 3/36$$

$$X = (1-2/30)(1-3/36)(1-7/8)$$
$$= .1069$$

C

7 This is a typical exam question on probability calculations. You need to write down expressions for the two items given in the problem. Then you should express them both in terms of l_x values. Then you need to discover how to solve for the item asked in the question.

Item I: ${}_{20}p_{20}\,({}_{20}p_{40}) = .733333 = (l_{40}/l_{20})(l_{60}/l_{40}) = l_{60}/l_{20}$

Item II: ${}_{10}q_{20} = 96/800 = 1 - {}_{10}p_{20} = 1 - l_{30}/l_{20}$

$$704/800 = l_{30}/l_{20}$$

The question asks for ${}_{30}q_{30} = 1 - {}_{30}p_{30} = 1 - l_{60}/l_{30}$.
The solution should be clear, since $\dfrac{l_{60}}{l_{30}} = \dfrac{l_{60}/l_{20}}{l_{30}/l_{20}}$

$$ {}_{30}p_{30} = \frac{l_{60}}{l_{30}} = \frac{.733333}{704/800} = .8333 $$

$$ {}_{30}q_{30} = 1 - .8333 = .1667 $$

E

8 The key to working this problem is knowing what the various symbols represent. It seems like very little work for a 4 point question.

$${}_np_{xx} = ({}_np_x)({}_np_x) = .25 \qquad \therefore {}_np_x = .50$$

$$\bar{X} = {}_nq_x + {}_nq_{xx} - {}_{n|}q_{xxx}$$

$$= (1 - {}_np_x) + (1 - {}_np_{xx}) - ({}_np_{xxx} - {}_{n+1}p_{xxx})$$

$$= 1 - .50 + 1 - .25 - {}_np_x({}_np_{xx})[1 - p_{x+n:x+n:x+n}]$$

$$= .50 + .75 - .25(.50)[1 - .50(.50)(.50)]$$

$$= .50 + .75 - .1094$$

$$= 1.1406 \qquad \text{B}$$

9 This is a typical exam question involving identities for annuities. What is not typical is that it is VERY hard to reduce this to a workable formula to solve for the interest rate. It took me WAY too long to work this!

$$\ddot{a}_{\overline{n+2}|} = 14.030 = \frac{1-v^{n+2}}{d} = \ddot{a}_{\overline{n}|} + v^n \ddot{a}_{\overline{2}|}$$

$$\ddot{s}_{\overline{n}|} = 52.344 = \frac{(1+i)^n - 1}{d} = \ddot{a}_{\overline{n}|}(1+i)^n = \frac{\ddot{a}_{\overline{n}|}}{v^n}$$

After many false starts, it appears the quickest solution uses substitution for $(1+i)^n$ from the second equation:

$$52.344 = \frac{(1+i)^n - 1}{d} \Rightarrow 1 + 52.344d = (1+i)^n$$

$$\ddot{a}_{\overline{n+2}|} = \ddot{a}_{\overline{n}|} + v^n \ddot{a}_{\overline{2}|}$$

$$14.030 = \frac{\ddot{s}_{\overline{n}|} + \ddot{a}_{\overline{2}|}}{(1+i)^n}$$

$$= \frac{52.344 + (1 + v^1)}{52.344d + 1}$$

$$14.030(52.344d + 1) = 52.344 + 1 + (1 - d)$$

$$734.386d + 14.030 = 54.344 - d$$

$$735.386d = 40.314$$

$$d = .0548$$

$$v = .9452$$

$$i = 5.80\%$$

E

10 This is a typical exam question on increasing and decreasing annuities. You should memorize the formulas for these, since they appear frequently on the exam:

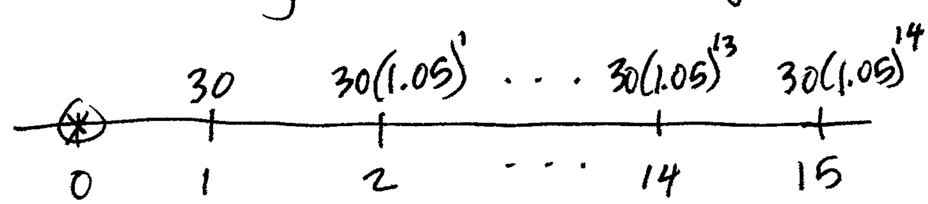
$$(Ia)_{\overline{n}|i} = a_{\overline{n}|} + 1\left(\frac{a_{\overline{n}|} - nv^n}{i}\right) = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i}$$

$$(Da)_{\overline{n}|i} = na_{\overline{n}|} - 1\left(\frac{a_{\overline{n}|} - nv^n}{i}\right) = \frac{n - a_{\overline{n}|i}}{i}$$

X is a 10 year decreasing annuity, where the payment decreases by \$10 each year.

$$\begin{aligned} X &= 10\,(Da)_{\overline{10}|.065} \\ &= 10\left[\frac{10 - a_{\overline{10}|.065}}{.065}\right] \\ &= 432.49 \end{aligned}$$

You need to write down the payments for the second annuity on a time line diagram:



$$Y = 30v^1 + \frac{30(1.05)^1}{(1.065)^2} + \cdots + \frac{30(1.05)^{14}}{(1.065)^{15}}$$

10 continued

The standard trick to evaluate a series of payments like this is to factor out the first term of the series. This allows the exponents in the numerator and denominator to "line up":

$$Y = \frac{30}{1.065}\left[\frac{1.05}{1.065} + \cdots + \left(\frac{1.05}{1.065}\right)^{14}\right]$$

$$= \frac{30}{1.065}\,\ddot{a}_{\overline{15}|k} \quad \text{where } 1+k = \frac{1.065}{1.05} = 1.0143$$

$$= \frac{30}{1.065}\,\ddot{a}_{\overline{15}|1.43\%}$$

$$= \left(\frac{30}{1.065}\right)(1.0143)\,a_{\overline{15}|1.43\%}$$

$$= 28.1690(1.0143)(13.4160)$$

$$= 383.32$$

$$|X - Y| = 432.49 - 383.32$$

$$= 49.17$$

E

11 This question tests the idea of mortality improvement, which has been tested many times in recent years. The correct interpretation of the improvement is that the projected q_x value after n years is given by $q_x(1-.015)^n$. It is incorrect to adjust the p_x value.

You should write out a table with the projected q_x values for the next few years:

Age x	q_x 2012	q_x 2013	q_x 2014
65	.0156	$.0156(.985) = .015366$	
66	.0176	$.0176(.985) = .017336$	$.0176(.985)^2 = .017076$

Smith's annuity has payments of 100,000 for two years. The payments occur at the end of each year (ages 66 and 67).

$$NPV = 100{,}000\left(v^1 p_{65} + v^2 p_{65} p_{66}\right)$$

$$= 100{,}000\left[\frac{1-.015366}{1.05} + \frac{(1-.015366)(1-.017076)}{(1.05)^2}\right]$$

$$= 100{,}000\,(.937747 + .877842)$$

$$= 181{,}559$$

C

12 This is a fairly complex multiple decrement problem. The basic idea has not been tested for several years, but this problem is quite similar to 2006 #28.

The problem tells you there is a two decrement table with mortality and withdrawal. You are told that mortality in the single decrement table has uniform distribution of decrements (UDD). The withdrawal in the single decrement table has a constant force of decrements.

The question asks for the value of $q_x^{(d)}$, which is the probability of mortality in the multiple decrement table. To determine this, you can use formula 10.6.3 from the Bowers book:

$$q_x^{(1)} = \int_0^1 {}_tp_x^{(T)} \mu_{x+t}^{(1)} \, dt = \int_0^1 {}_tp_x'^{(1)} \, {}_tp_x'^{(2)} \mu_{x+t}^{(1)} \, dt$$

The key to working this problem is twofold. First, you must be comfortable evaluating the resulting integral, which is fairly rare on EA-1 exam problems. The second key to the problem is that the UDD assumption means that ${}_tp_x'^{(d)} \mu_{x+t}^{(d)} = q_x'^{(d)}$. This is what allows you to evaluate the integral:

$$q_x^{(d)} = \int_0^1 {}_tp_x'^{(w)} \, {}_tp_x'^{(d)} \mu_{x+t}^{(d)} \, dt$$
$$= q_x'^{(d)} \int_0^1 {}_tp_x'^{(w)} \, dt$$

(12) continued

$$q_x^{(d)} = .03 \int_0^1 {}_tp_x^{\prime(w)}\,dt$$

Now you can use the constant force of withdrawal to evaluate the integral. The relationship between probability and force for a decrement is

$${}_np_x = e^{-\int_x^{x+n} \mu_t\,dt}$$

$${}_tp_x^{\prime(w)} = e^{-\int_0^t \mu_y\,dy} = e^{-.20t}$$

$$q_x^{(d)} = .03 \int_0^1 e^{-.20t}\,dt$$

$$= .03 \int_0^1 e^{-t/5}\,dt$$

$$= .03\left(-5e^{-t/5}\right)\Big|_0^1$$

$$= .03(-5)(1-e^{-1/5})$$

$$= .02719$$

B

Evaluating the integral requires knowledge of some calculus. Here are the indefinite integrals that I used to get the final result

$$\int e^t = e^t \qquad \int e^{-t} = -e^{-t} \qquad \int e^{-t/5} = -5e^{-t/5}$$

$$e^t = \frac{d}{dt}e^t \qquad e^{-t} = \frac{d}{dt}(-e^{-t}) \qquad e^{-\frac{t}{5}} = \frac{d}{dt}(-5e^{-t/5})$$

13 This is a typical exam question on yield curves and spot rates. If you know the definition of a spot rate, the calculation is quite simple. Each spot rate corresponds to the yield on a zero coupon bond of the same duration.

You need to calculate the seven year annuity due factor based on the spot rates. This uses six different interest rates:

$$\ddot{a}_{\overline{7}|j,k,\dots} = 1 + (1.0619)^{-1} + (1.0732)^{-2} + (1.0783)^{-3} + (1.0803)^{-4} + (1.0818)^{-5} + (1.0833)^{-6}$$
$$= 5.6354$$

$$X = \frac{1,000,000}{5.6354}$$
$$= 177,449$$

B

14 This is NOT a typical exam question on multiple decrements. One of the things that makes this problem difficult is the unusually imprecise wording.

The problem asks for the "organizations rate of mortality at age 62". This sounds like the rate of mortality in a single decrement table, but you aren't given enough information to work the problem that way.

I think the problem is really asking for the central death rate, or the central rate of decrement. For a single decrement table, you calculate the central death rate as

$$m_x = d_x / L_x$$

where L_x is the mean value of l_x from age x to age $x+1$. With multiple decrements, you can use a similar approach. First, write down all the changes in the population from 62 to 63:

	-3 deaths	Disabled -5	Term -10	New 15	Ret'd -4	
01-01	06-01	07-01	08-01	10-01	11-01	12-31
125 lives	122	117	107	122	118	118

(14) continued

Now you can calculate the average value of l_x from 62 to 63:

$$L_{62} = \frac{5}{12}(125) + \frac{1}{12}(122) + \frac{1}{12}(117) + \frac{2}{12}(107) + \frac{1}{12}(122) + \frac{2}{12}(118)$$

$$= 119.6667$$

$$d_{62} = 3$$

$$m_{62} = 3/119.6667$$
$$= .02507$$

A

When I first worked this problem, I calculated the probability of mortality, which is 3/125 or .02400. This is also in answer range A, but it is outside the "implied range" – which is values from .0250 to .0255.

15 This is a typical exam question on actuarially equivalent benefits. Since these benefits are actuarially equivalent, they both have the same present value. In some earlier exam problems, you end up with multiple unknowns and must use a bit of algebra to get the final answer. This one is more straightforward.

PV of I: $30{,}000\,\ddot{a}_{62}$

PV of II: $50{,}000 + v^3\,{}_3p_{62}\left(X\,\ddot{a}_{\overline{5}|.07}\right)$

$$30{,}000\,\ddot{a}_{62} = 50{,}000 + (1.07)^{-3}\,{}_3p_{62}\left(X\,\ddot{a}_{\overline{5}|.07}\right)$$

$$X = \frac{(30{,}000\,\ddot{a}_{62} - 50{,}000)(1.07)^3}{{}_3p_{62}\left(\ddot{a}_{\overline{5}|.07}\right)}$$

$$= \frac{(30{,}000(12.68) - 50{,}000)1.225}{(.99)^3(1.07)(a_{\overline{5}|.07})}$$

$$= \frac{404{,}754}{4.2569}$$

$$= 95{,}082$$

E

16 This is a typical exam question on probability calculations. The key is being able to interpret the symbol given for X.

In general, $_{n|m}q_x$ is the probability that x lives for n years, then dies within the next m years:

$$X = {}_{.25|1.50}q_x = {}_{.25}p_x - {}_{1.75}p_x$$
$$= {}_{.25}p_x - p_x({}_{.75}p_{x+1})$$
$$= (1 - {}_{.25}q_x) - (1-q_x)(1 - {}_{.75}q_{x+1})$$

You are told that deaths correspond to the uniform distribution of decrements assumption. Under UDD, you know that ${}_tq_x = t(q_x)$.

$$X = (1 - .25(q_x)) - (1-q_x)(1 - .75(q_{x+1}))$$
$$= [1 - .25(.03)] - (1-.03)[1 - .75(.04)]$$
$$= .9925 - .97(.97)$$
$$= .0516$$

C

There is one feature of this calculation that is a bit fishy. Based on the q_x values given, the answer can only have two significant decimal places. But that is insufficient to give the correct answer. In effect you must assume that $q_x = .03000$ and $q_{x+1} = .04000$ (to get more than 2 decimal places).

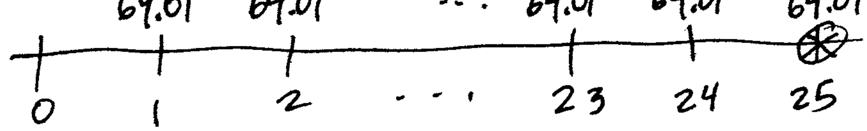
17 This question is somewhat like a sinking fund problem. You have to determine the loan payment based on the 4% per annum interest:

Loan payment $= 1{,}000 / a_{\overline{25}|4\%}$
$= 64.01$

The loan payments accumulate in a new fund at a nominal rate of 5.0%, compounded semiannually. You must convert this to an annual effective rate:

$$1+j = \left(1+\frac{.05}{2}\right)^2 = 1.050625 \Rightarrow j = 5.0625\%$$

The questions asks for Jones' annual yield over the entire 25 years. Jones starts with 1000 at time zero. You need to calculate the accumulated fund value at the end of 25 years:



Accumulated value at end of 25 years is $64.01\, s_{\overline{25}|5.0625\%}$. Now equate this to the starting value of 1000:

$$1000(1+x)^{25} = 64.01(48.14) = 3081.56$$
$$1+x = (3.08156)^{1/25} \Rightarrow x = 4.60\%$$

B

18 This is a straightforward question on benefits that are actuarially equivalent. Both Option 1 and Option 2 have the same present value:

PV of 1: $12(1000)\left[\ddot{a}^{(12)}_{\overline{5}|5\%} + {}_{5|}\ddot{a}^{(12)}_{60}\right]$

PV of 2: $12X\left[{}_{5|}\ddot{a}^{(12)}_{60}\right]$

You need to calculate the monthly interest rate equivalent to the annual rate of 5%:

$$(1+j)^{12} = 1.05 \Rightarrow j = .4074\%$$

$$\ddot{a}^{(12)}_{\overline{5}|.05} = \frac{1}{12}\left(\ddot{a}_{\overline{60}|.4074\%}\right)$$

$$= \frac{1}{12}(1.004074)\,a_{\overline{60}|.4074\%}$$

$$= 4.4459$$

$$12X\left[{}_{5|}\ddot{a}^{(12)}_{60}\right] = 12(1000)\left[4.4459 + {}_{5|}\ddot{a}^{(12)}_{60}\right]$$

$$X = 1000\left(\frac{4.4459 + 8.88}{8.88}\right)$$

$$= 1500.66$$

C

19 This is a typical question on calculation of the regular duration. Duration is a measure of the average weighted time of payment for a series of payments.

regular duration (Macaulay duration) $\bar{d} = \dfrac{\sum t P_t v^t}{\sum P_t v^t}$

The weight for each payment is the present value of each payment. For this bond, you need to calculate the quarterly interest that is equivalent to the annual nominal rate of 4.5%, compounded semiannually:

$$(1+j)^4 = \left(1+\frac{i^{(2)}}{2}\right)^2 = \left(1+\frac{.045}{2}\right)^2 = 1.045506$$

$$j = (1.045506)^{1/4} - 1 = 1.1119\% \text{ per quarter}$$

The bond has \$1000 face value with 4% coupons. Each quarterly coupon is \$10, for a period of 20 years (or 80 quarters).

$$\bar{d} = \left(\frac{1}{4}\right)\left[\frac{10(1v^1+2v^2+3v^3+\ldots+80v^{80}) + 80(1000v^{80})}{10(v^1+v^2+\ldots+v^{80}) + 1000v^{80}}\right]$$

Note the factor of 1/4. This is needed because the time periods for coupons are quarters instead of years.

(19) continued

$$\bar{d} = \frac{1}{4}\left(\frac{10\, Ia_{\overline{80}|1.1119\%} + 80(1000)v^{80}}{10\, a_{\overline{80}|1.1119\%} + 1000v^{80}}\right)$$

You should memorize the formulas for $Ia_{\overline{n}|i}$ and $Da_{\overline{n}|i}$, since they will save you time when you work these types of problems on the exam.

$$Ia_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} = a_{\overline{n}|} + \frac{a_{\overline{n}|} - nv^n}{i}$$

$$Ia_{\overline{80}|1.1119\%} = a_{\overline{80}|} + \frac{a_{\overline{80}|} - 80(1.01119)^{-80}}{1.1119\%}$$

$$a_{\overline{80}|} = 52.680$$

$$Ia_{\overline{80}|} = 1825.066 = 52.680 + \frac{52.680 - 32.852}{.01119}$$

$$\bar{d} = \frac{1}{4}\left(\frac{10(1825.066) + 1000(32.852)}{10(52.680) + 1000(1.01119)^{-80}}\right)$$

$$= \frac{1}{4}\left(\frac{18,251 + 32,852}{526.80 + 410.65}\right)$$

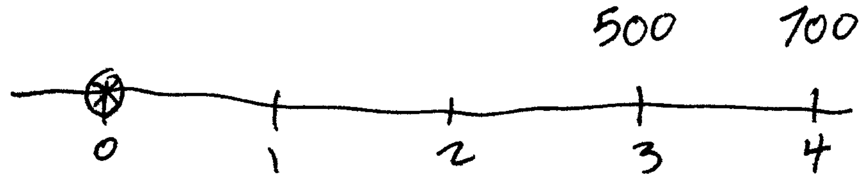
$$= 13.628$$

E

If you left out the factor of 1/4, your duration would be four times ~~too~~ big. Based on the answer ranges, it would be clear the result was incorrect.

20 This is a typical exam question on yield curves and spot rates. You must know the definition of a spot rate to work this problem – each spot rate corresponds to the yield on a zero coupon bond of the same duration.

You should write down the payouts on a time line diagram



The problem does not give you the 3 year spot rate, which would be used to discount the payout of $500. Let the 3 year spot rate be represented by Y. Then you can write the equation of value for this investment:

$$950 = 500(1+Y)^{-3} + 700(1.07)^{-4}$$

What exactly is represented by X? It is described as a three year deferred, one year spot rate. Another description is that it equals the forward rate from year 3 to year 4:

$$(1+Y)^{-3}(1+X)^{-1} = (1.07)^{-4}$$

(20) continued

That interpretation of X is the real trick to the problem. Now you can determine the value of X. First you must calculate the value of Y:

$$950 = 500(1+Y)^{-3} + 700(1.07)^{-4}$$

$$500(1+Y)^{-3} = -700(1.07)^{-4} + 950$$
$$(1+Y)^{-3} = (950 - 534.03)/500 = .8319$$

Now you can go back to the expression with both X and Y

$$(1+Y)^{-3}(1+X)^{-1} = (1.07)^{-4}$$

$$(1+X) = (1.07)^{4} * (1+Y)^{-3}$$
$$= (1.3108)(.8319)$$

$$X = 9.05\%$$

E

21 This is the first EA-1 question on the idea of a one-year term cost. The key to working this problem is being able to correctly guess what this means (unless you already know the definition). The one year term cost is the present value of benefits for exits expected to occur within the one year period from the valuation date.

The data given does not clearly define lower case x as the age - but that is the only reasonable interpretation. You need to calculate the one year term cost for participants at ages 30, 35 and 40. Instead of telling you the number of participants you are given the total salary for participants at each age.

The termination benefit is a lump sum of 50% of pay. The terminations are assumed to occur in the middle of the year, and the interest rate is 8%:

$$X = (1.08)^{-6/12}\left[\begin{array}{l} .15(5{,}000{,}000)(.50) \\ +\ .10(9{,}000{,}000)(.50) \\ +\ .05(6{,}000{,}000)(.50) \end{array}\right]$$

$$= .96225\,[375{,}000 + 450{,}000 + 150{,}000]$$

$$= 938{,}194 \qquad \text{(B)}$$

Simple interest gives a result of 937,500, also in (B)

22. This is a typical exam question on insurance values. You need to know the relationship between A_x and A_{x+1}, or you need to be able to derive it. In addition, you must know the definition of $_nE_x$: $_nE_x = v^n {}_np_x$.

$$A_x = vq_x + v^2 {}_{1|}q_x + v^3 {}_{2|}q_x + \dots$$

$$A_{x+1} = vq_{x+1} + v^2 {}_{1|}q_{x+1} + \dots$$

$$A_x - A_{x+1}(vp_x) = vq_x = v(1-p_x)$$

$$A_x - A_{x+1}({}_1E_x) = v - {}_1E_x$$

$$A_x - v + {}_1E_x = A_{x+1}({}_1E_x)$$

$$\frac{A_x - v}{{}_1E_x} + 1.0 = A_{x+1}$$

$$A_{77} = \frac{A_{76} - v}{{}_1E_{76}} + 1.0$$

$$= \frac{.80 - (1.03)^{-1}}{.90} + 1.0$$

$$= .81$$

C

You can derive the relationship between insurance values at successive ages by writing A_x with l_x, d_x:

$$A_x = (vd_x + v^2 d_{x+1} + v^3 d_{x+2} + \dots)/l_x$$

$$A_{x+1} = (vd_{x+1} + v^2 d_{x+2} + v^3 d_{x+3} + \dots)/l_{x+1}$$

23 One point of this problem is NOT being scared of the words "survival model". This is simply a formula for l_x values. The key to this problem is correctly writing the formula for the probability the question asks for:

$_{1|}q_{\overline{2:3}}$ is the probability that the joint lives currently age 2 and 3 survive one year, and die in the next year

$$p_{\overline{xy}} = p_x + p_y - p_{xy}$$

$$q_{\overline{xy}} = q_x + q_y - q_{xy}$$

$$_{1|}q_{\overline{2:3}} = {_{1|}q_2} + {_{1|}q_3} - {_{1|}q_{2:3}}$$

$$= ({_1p_2} - {_2p_2}) + ({_1p_3} - {_2p_3}) - ({_1p_{2:3}} - {_2p_{2:3}})$$
$$= \quad '' \quad + \quad '' \quad - [p_2(p_3) - {_2p_2}({_2p_3})]$$

Since you are given a formula for l_x values, you should express the equation in terms of l_x values

$$_{1|}q_{\overline{2:3}} = \left(\frac{l_3 - l_4}{l_2}\right) + \left(\frac{l_4 - l_5}{l_3}\right) - \left[\frac{l_4}{l_2} - \frac{l_4}{l_2}\left(\frac{l_5}{l_3}\right)\right]$$

$l_2 = (5-2)^2 = 9$
$l_3 = (5-3)^2 = 4$
$l_4 = (5-4)^2 = 1$ and $l_5 =$ zero

(23) continued

Now you can plug those values into the prior equation

$$1|q_{\overline{2:3}} = \frac{4-1}{9} + \frac{1-0}{4} - \left[\frac{1}{9} - \frac{1}{9}\left(\frac{0}{4}\right)\right]$$

$$= \frac{3}{9} + \frac{1}{4} - \frac{1}{9}$$

$$= .472$$

D

24 This type of question on projected mortality improvements has become very popular on the EA-1 exam. The correct interpretation of the improvement is that the projected q_x value after n years is given by $q_x(1-.02)^n$.

You should write out a table with the projected q_x values for the next few years - and ~~then~~ for ten years later

Age x	q_x 2013	q_x 2014	q_x 2015	q_x 2023	q_x 2024	q_x 2025
50	.0148			$*(.98)^{10}$		
51	.0159	$*(.98)^1$	$*(.98)^2$	* "	$*(.98)^{11}$	
52	.0170	$*(.98)^1$	$*(.98)^2$	* "	* "	$*(.98)^{12}$

Now you should write out the formulas for X and Y, which use different mortality rates (10 years apart).

$X = 10{,}000(1 + v\,p_{50} + v^2\,{}_2p_{50})$ at 01/01/2013

$= 10{,}000\left[1 + \dfrac{1-.0148}{1.04} + \dfrac{(1-.0148)(1-.0159(.98))}{(1.04)^2}\right]$

$= 10{,}000[1 + .94731 + .94731(.94656)]$

$= 28{,}440$

$Y = 10{,}000(1 + v\,p_{50} + v^2\,{}_2p_{50})$ at 01/01/2023

$= 10{,}000\left[1 + \dfrac{(1-.0148(.98)^{10})}{1.04} + \dfrac{(1-.0148(.98)^{10})(1-.0159(.98)^{11})}{(1.04)^2}\right]$

(24) continued

$$Y = 10{,}000\,[\,1 + .94991 + .94991(.94930)\,]$$
$$= 28{,}517$$

$$|X - Y| = 28{,}517 - 28{,}440$$
$$= 76.71$$

B

Once again, this problem violates the concept of significant digits. The q_x values given only have three significant digits, so the values of X and Y can't really have five significant digits – but you have no valid answer to the problem unless you carry out the decimal places as shown in the calculation above.

25 This is a typical exam question on loan amortization schedules. The key to working this problem is writing down the amortization schedule.

The original loan is X, with payments for n years at 12.5% interest. Let P be the annual loan payment, which is $X/a_{\overline{n}|}$, $X = P \cdot a_{\overline{n}|}$.

Payment #	Pmt	Interest paid	Principal paid	O/S Loan
1	P	$P(1-v^n)$	Pv^n	$P a_{\overline{n-1}|}$
2	P	$P(1-v^{n-1})$	Pv^{n-1}	$P a_{\overline{n-2}|}$
⋮	⋮	⋮	⋮	⋮
n-1	P	$P(1-v^2)$	Pv^2	$P a_{\overline{1}|}$
n	P	$P(1-v^1)$	Pv^1	zero

The interest portion of the final payment is 153.86, and the principal repaid through the first n-1 payments is 6009.12. You have to read the data carefully - it says "total principal repaid".

$P a_{\overline{1}|} = X - 6009.12$ O/S loan after n-1 payments

$153.86 = (X - 6009.12).125$ final interest payment

$X = 7240$ original loan amount

(25) continued

$153.86 = P(1-v^1)$ final interest payment

$\frac{153.86}{1-(1.125)^{-1}} = P = 1384.74$

Principal repaid in first payment is Pv^n. But you don't need to solve for n. Instead you can subtract the loan interest in the first year from the payment P:

$Y = P -$ interest paid in 1^{st} year

$= P - 7240(.125)$

$= 1384.74 - 905.00$

$= 479.74$

(C)

If you do solve for the annuity, the resulting loan amortization period is 9 years:

$P = 1384.74 = 7240 / a_{\overline{n}|12.5\%}$

$a_{\overline{n}|.125} = 5.2284 \Rightarrow n = 9$ years

25 This is an interesting idea - considering the impact of default on the price of a bond. The bond itself is very simple, since you receive a single coupon plus redemption of the face amount at the end of the year.

$$P = X\%(\text{zero}) + (1 - X\%)(1000 + 100)v^1$$

$$980 = (1 - X\%)(1100/1.07)$$

$$1 - X\% = .9533$$

$$X = 4.67\%$$

E

27 This is a simplified question on multiple decrement tables. The key point is that decrement 2 occurs at the end of the year.

You can calculate how many lives remain after being exposed to both decrements 1 and 3. In effect you are treating this like a double decrement table:

$$\begin{aligned} {}_tp_x^{(\tau)} &= {}_tp_x'^{(1)} \cdot {}_tp_x'^{(3)} \\ &= (1-{}_tq_x'^{(1)})(1-{}_tq_x'^{(3)}) \\ &= (1-.05)(1-.30) \\ &= .95(.70) = .665 \end{aligned}$$

Of the 1000 lives at age x, 665 will remain at the end of the year - prior to the effect of decrement 2. The number of exits for decrement 2 is 19.95:

$$\begin{aligned} d_x^{(2)} &= 665(.03) \\ &= 19.95 \end{aligned}$$

B

28 This is a typical exam question on actuarially equivalent joint and survivor annuities. You need to write down formulas for the present value of each annuity (which are all equal):

Annuity I $PV = 12(1250)\ddot{a}_x^{(12)}$

" II $PV = 12(1000)\left[\ddot{a}_x^{(12)} + 50\%\left(\ddot{a}_y^{(12)} - \ddot{a}_{xy}^{(12)}\right)\right]$

" III $PV = 12(X)\left[\ddot{a}_x^{(12)} + 75\%\left(\ddot{a}_y^{(12)} - \ddot{a}_{xy}^{(12)}\right)\right]$

This is the usual situation in these problems. When you have no present value factors, there must be an algebra trick to get rid of all the unknowns. The key is to divide both sides of the equations by $\ddot{a}_x^{(12)}$:

$$12(1250)\ddot{a}_x^{(12)} = 12(1000)\left[\ddot{a}_x^{(12)} + 50\%\left(\ddot{a}_y^{(12)} - \ddot{a}_{xy}^{(12)}\right)\right]$$

$$12(1250)\ddot{a}_x^{(12)} = 12X\left[\ddot{a}_x^{(12)} + 75\%\left(\ddot{a}_y^{(12)} - \ddot{a}_{xy}^{(12)}\right)\right]$$

$$1250 = 1000\left[1 + .50\left(\ddot{a}_y^{(12)} - \ddot{a}_{xy}^{(12)}\right)/\ddot{a}_x^{(12)}\right]$$

The trick is to replace the ratio of annuity values with a single unknown R:

$$1250 = 1000[1 + .5R]$$

(28) continued

You can do the same thing with the 2nd equation:

$$12(1250)\,\ddot{a}_x^{(12)} = 12X\left[\ddot{a}_x^{(12)} + 75\%\left(\ddot{a}_y^{(12)} - \ddot{a}_{xy}^{(12)}\right)\right]$$

$$1250 = X\left[1 + .75R\right]$$

Use the first equation to calculate the value of R. Then you can substitute that in the above equation to calculate a value for X:

$$1250 = 1000\left[1 + .5R\right]$$
$$1.25 = 1 + .5R$$
$$.50 = R$$

$$1250 = X\left[1 + .75(.50)\right]$$
$$\frac{1250}{1.375} = X \Rightarrow X = 909.09$$

E

29 This is a fairly typical exam question on perpetuities. You have a perpetuity where the payments increase every other year. You should write an expression for the present value, then see if you can simplify the calculation:

$$X = 1(1.06)^{-2} + 2(1.06)^{-4} + 3(1.06)^{-6} + \dots$$

Now multiply both sides by $(1.06)^{-2}$ and subtract:

$$X(1.06)^{-2} = \qquad 1(1.06)^{-4} + 2(1.06)^{-6} + \dots$$

$$X(1-(1.06)^{-2}) = 1(1.06)^{-2} + 1(1.06)^{-4} + 1(1.06)^{-6} + \dots$$

The right hand side of this equation is a perpetuity. You can evaluate this using the standard formula for a perpetuity immediate $\Rightarrow PV = 1/i$, but you must determine the equivalent biennial interest rate.

Let $1+j = (1.06)^2 = 1.1236 \Rightarrow j = 12.36\%$ every 2 years

$$X(1-(1.1236)^{-1}) = 1/j$$
$$X(1-.8900) = 1/.1236$$
$$X = 8.0906/.1100$$
$$= 73.55$$

B

30 This is an interesting question on the expectation of life. You must know the definition of e_x

$$e_x = p_x + {}_2p_x + {}_3p_x + \ldots$$

$$e_0 = p_0 + {}_2p_0 + {}_3p_0 + \ldots$$

$$= \frac{l_1 + l_2 + l_3 + \ldots}{l_0}$$

You are given values for q_x for various age ranges. You need to determine the p_x values too:

	Ages $0 \le x \le 35$	Ages $36 \le x \le 75$	Ages $x \ge 76$
q_x	$.01/1.01$	$.02/1.02$	1.00
p_x	$1 - q_x$	$1 - q_x$	zero
	$= 1/.01$	$= 1/1.02$	
	$= .9901$	$= .9804$	

Now you can construct the formula to calculate e_0. You will have one series of terms involving powers of .9901, and another series with both powers of .9901 and .9804:

$$e_0 = (.9901)^1 + (.9901)^2 + \ldots + (.9901)^{35} + (.9901)^{36}(1 + (.9804)^1 + \ldots]$$

There are thirty five terms which represent the probability of survival to age 35. The next term is survival to age 36. It is multiplied by a series of terms for probability of survival after 36.

(30) continued

$$e_0 = (.9901)^1 + \ldots + (.9901)^{35} \quad \text{35 terms}$$
$$+ (.9901)^{36}\left(1 + (.9804)^1 + \ldots + (.9804)^{40}\right) \quad \text{41 terms}$$

You need to count carefully to make sure you have the correct number of terms. There should be nothing for ages 77 and later. Once the participant survives to age 76, there are no further terms. This is because the probability of survival after 76 is zero.

You can evaluate the series using annuity calculations at two different interest rates - and you should be able to guess what the rates are.

$$e_0 = a_{\overline{35}|j} + (.9901)^{36}(1 + a_{\overline{40}|k})$$

$(1+j)^{-1} = .9901$ $\quad$ $(1+k)^{-1} = .9804$

$j = 1.0\%$ $\quad$ $k = 2.0\%$

$$e_0 = a_{\overline{35}|1.0\%} + (.9901)^{36}(1 + a_{\overline{40}|2.0\%})$$
$$= 29.4086 + .6989(1 + 27.3555)$$
$$= 49.227$$

(B)

31 This is a typical exam question on joint and survivor annuities and actuarially equivalent benefits. The difference versus other problems on the exam is that you have various factors given in the problem data. Instead of using an algebra trick, here you simply do some ugly arithmetic.

You need to write down formulas for the present values of both annuities. The second one is potentially tricky, but easy if you know how to use reversionary annuities.

PV of I: $10{,}000\ddot{a}_{65}$

PV of II:

$X(\ddot{a}_{65:62})$ while both alive

$+.75X(\ddot{a}_{65}-\ddot{a}_{65:62})$ Smith after Jones dies

$+.75X(\ddot{a}_{62}-\ddot{a}_{65:62})$ Jones after Smith dies

$$10{,}000\ddot{a}_{65} = X(\ddot{a}_{65:62}) + .75X[\ddot{a}_{65}+\ddot{a}_{62}-2(\ddot{a}_{65:62})]$$

$$X = 10{,}000\ddot{a}_{65}/[\ddot{a}_{65:62}+.75(\ddot{a}_{65}+\ddot{a}_{62}-2(\ddot{a}_{65:62}))]$$

$$= 10{,}000/\left[\frac{\ddot{a}_{65:62}}{\ddot{a}_{65}}+.75\left(1+\frac{\ddot{a}_{62}-2(\ddot{a}_{65:62})}{\ddot{a}_{65}}\right)\right]$$

$$= 10{,}000/[8.7060/10.0426+.75(.3314)]$$

$$= 8{,}965$$

D

32 This is a typical exam question on calculation of the dollar weighted and time weighted rates of return. The key to working this problem is writing down all the cash flows and market values on a time line diagram.

For the time weighted returns, you need to know the market values before and after each cash flow. I'll use MV_A to indicate the market value after the cash flow, and MV_B for the market value before the cash flow.

date	01-01	4-1	10-1	12-31
MV_B		110,000	110,000	180,000
cashflow		-20,000	75,000	0
MV_A	100,000	90,000	185,000	180,000

The time weighted return is calculated by using ratios of market values between the cash flows. This measures the change in assets, without including the effect of the cash flow:

$$1+t = \underbrace{\frac{110{,}000}{100{,}000}}_{\text{1st 3 months}}\underbrace{\left(\frac{110{,}000}{90{,}000}\right)}_{\text{next 6 months}}\underbrace{\left(\frac{180{,}000}{185{,}000}\right)}_{\text{last 3 months}}$$

$$= 1.3081 \Rightarrow t = 30.81\%$$

(32) continued

The dollar weighted return calculation reflects all the cash flows, as well as the time periods. In general, you should use simple interest for fractions of a year:

$$100{,}000\left(1+\frac{12}{12}d\right) - 20{,}000\left(1+\frac{9}{12}d\right) + 75{,}000\left(1+\frac{3}{12}d\right) = 180{,}000$$

Now group all terms involving d on one side, and everything else on the other side of the equation

$$\left[100{,}000\left(\tfrac{12}{12}\right) - 20{,}000\left(\tfrac{9}{12}\right) + 75{,}000\left(\tfrac{3}{12}\right)\right]d = 180{,}000 - 155{,}000$$

$$d = 25{,}000 / \left[100{,}000 - 20{,}000\left(\tfrac{9}{12}\right) + 75{,}000\left(\tfrac{3}{12}\right)\right]$$
$$= 24.10\%$$

The last calculation assumes the cash flows occur at mid-year. You can modify the final fraction for the dollar weighted return to calculate m - just use 6/12 for the weight on the last two cash flows:

$$m = 25{,}000 / \left[100{,}000 - 20{,}000\left(\tfrac{6}{12}\right) + 75{,}000\left(\tfrac{6}{12}\right)\right]$$
$$= 19.61\%$$

$t > d > m$ (B)

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