JOINT LIFE STATUS

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p_{XY} = Probability that two lives age x and y <u>both</u> live one year

$= p_X(p_Y)$

$p_{\overline{XY}}$ = Probability that <u>at least one</u> of the two lives age x and y live one year

$= p_X + p_Y - p_{XY}$

$= 1 - (1-p_X)(1-p_Y)$

$\ddot{a}_{XY}$ = Annuity payable while two lives age x and y are <u>both</u> alive

$= 1 + vp_{XY} + v^2{}_2p_{XY} + \ldots$

$\ddot{a}_{\overline{XY}}$ = Annuity payable while <u>at least one</u> of the two lives age x and y live

$= \ddot{a}_X + \ddot{a}_Y - \ddot{a}_{XY}$

REVERSIONARY ANNUITY

This pays \$1 annually to life x after the death of life y:

$$\ddot{a}_{Y|X} = \ddot{a}_X - \ddot{a}_{XY}$$

As long as x and y are both alive, this annuity pays \$1-\$1, or zero.

If x is alive and y is dead, the first annuity pays \$1, and the second pays 0.

<u>Pop Quiz</u>

Three lives are ages x, y and z. Write an expression for the PV of an annuity that pays \$500 per annum as long as <u>exactly</u> 2 of the three are alive.

JOINT AND SURVIVOR FACTOR

These formulas are for a benefit that reduces <u>only</u> upon employee's death

Assume life annuity of \$1 that is actuarially equivalent to J&S annuity of \$B, continues at rate K to beneficiary

$$1\,(\ddot{a}_X) = B\,(\ddot{a}_X + K(\ddot{a}_Y - \ddot{a}_{XY}))$$

$$B = \ddot{a}_X \;/\; (\ddot{a}_X + K(\ddot{a}_Y - \ddot{a}_{XY}))$$

$$B = 1 \;/\; \left(1 + K\frac{(\ddot{a}_Y - \ddot{a}_{XY})}{\ddot{a}_X}\right)$$

$$\text{If } K = 50\%,\quad B = (1 + .5[(\ddot{a}_Y - \ddot{a}_{XY}) / \ddot{a}_X])^{-1}$$

$$\text{If } K = 100\%,\quad B = (1 + 1.0[(\ddot{a}_Y - \ddot{a}_{XY}) / \ddot{a}_X])^{-1}$$

Given a J&S factor based on one continuation fraction (K), you can derive the J&S factor for any other value of K

JOINT AND SURVIVOR FACTOR

These formulas are for a benefit that reduces upon either the employee's death or joint annuitant's death

Assume life annuity of \$1 that is actuarially equivalent to J&S annuity of \$B, continues at rate K to survivor

$$1\,(\ddot{a}_X) = B\,(K\ddot{a}_X + K\ddot{a}_Y + (1-2K)\ddot{a}_{XY})$$

$$B = \ddot{a}_X \,/\, (K\ddot{a}_X + K\ddot{a}_Y + (1-2K)\ddot{a}_{XY})$$

- Can you derive this formula?
  Hint - use reversionary annuity concept

- This J&S factor B may be > 1.00

- Unlike other J&S annuity, can't solve for factors based on other values of K

JOINT AND SURVIVOR FACTORS

Comparison of J&S benefit types

For 100% continuation:

Reduce eedeath $= \ddot{a}_X + K(\ddot{a}_Y - \ddot{a}_{XY})$

$= \ddot{a}_X + \ddot{a}_Y - \ddot{a}_{XY}$

Reduce either dies $= K\ddot{a}_X + K\ddot{a}_Y + (1-2K)\ddot{a}_{XY}$

$= \ddot{a}_X + \ddot{a}_Y - \ddot{a}_{XY}$

For 50% continuation:

Reduce ee death $= \ddot{a}_X + K(\ddot{a}_Y - \ddot{a}_{XY})$

$= \ddot{a}_X + .5(\ddot{a}_Y - \ddot{a}_{XY})$

Reduce either dies $= K\ddot{a}_X + K\ddot{a}_Y + (1-2K)\ddot{a}_{XY}$

$= .5\ddot{a}_X + .5\ddot{a}_Y$

Verify results for all three cases -
Both X+Y alive, only X alive, only Y alive

JOINT AND SURVIVOR WITH POP-UP

Benefit reduces only upon employee's death. If beneficiary dies first, then the employee's benefit is restored to life annuity amount, as if no J&S option was ever elected.

Construct pieces of J&S pop-up benefit:

$B\,\ddot{a}_X$	benefit for life of ee
$+\,KB\,(\ddot{a}_Y - \ddot{a}_{XY})$	benefit if ee dies first
$+(1-B)(\ddot{a}_X - \ddot{a}_{XY})$	benefit if benef. dies first

$$
\begin{aligned}
1(\ddot{a}_X) &= B\,\ddot{a}_X + KB\ddot{a}_Y - KB\ddot{a}_{XY} \\
&\quad + \ddot{a}_X \qquad\qquad - \ddot{a}_{XY} \\
&\quad - B\,\ddot{a}_X \qquad\qquad + B\ddot{a}_{XY} \\
\ddot{a}_{XY} &= KB\ddot{a}_Y + (1-K)B\ddot{a}_{XY} \\
B &= \ddot{a}_{XY} / (K\ddot{a}_Y + (1-K)\ddot{a}_{XY})
\end{aligned}
$$

CERTAIN AND LIFE ANNUITY

Death benefit in first n years is continuation of benefits to end of n years.

Assume reduced benefit B is actuarially equivalent to life annuity of $1 per year:

$$1\,(\ddot{a}_x) = B\,(\ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x)$$

$$= B\,\left(\ddot{a}_{\overline{n}|} + \frac{D_{x+n}}{D_x}\,\ddot{a}_{x+n}\right)$$

$$= B\,(\ddot{a}_{\overline{n}|} + v^n\,{}_np_x\,\ddot{a}_{x+n})$$

Traditional definition:

$$B = \ddot{a}_x / (\ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x)$$

ACTUARIALLY REDUCED EARLY RETIREMENT BENEFITS

Assume reduced benefit B payable at age X is actuarially equivalent to annual life annuity of $1 commencing at RA:

$$1 \left(\frac{D_{RA}}{D_X}\right) \ddot{a}_{RA} = B\,(\ddot{a}_X)$$

$$B = \frac{D_{RA}\,\ddot{a}_{RA}}{D_X\,\ddot{a}_X}$$

$$= N_{RA} / N_X$$

$$= v^{RA-X}\,{}_{RA-X}p_X\,(\ddot{a}_{RA} / \ddot{a}_X)$$

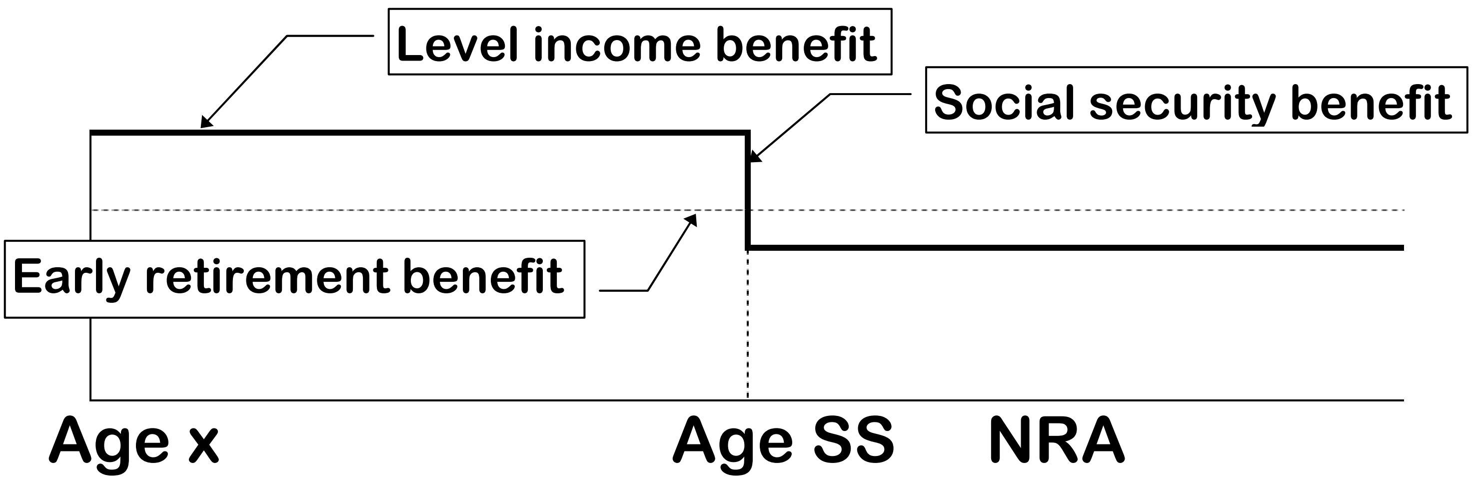
Actuarial reduction consists of three components that reflect effect of early commencement of benefits:

1. **Interest lost on pension fund assets**
2. **Lost mortality gains**
3. **More payments to annuitant**

LEVEL INCOME BENEFITS

Provide benefit from pension plan that is

1. Non-level
2. Actuarially equivalent to normal form
3. Produces level benefit when added to expected Social Security benefit



$$ERB\,\ddot{a}_x = LIB * \ddot{a}_{x:\overline{ss-x}|} + (LIB - PIA) * {}_{ss-x|}\ddot{a}_x$$

$$= LIB(\ddot{a}_x) - PIA\left(\frac{D_{ss}\,\ddot{a}_{ss}}{D_x}\right)$$

$$LIB = ERB + PIA\left(\frac{D_{ss}\,\ddot{a}_{ss}}{D_x\,\ddot{a}_x}\right)$$

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